Nihonkai Math. J.
Vol. 12(2001), 59-70

OPERATOR INEQUALITIES RELATED TO THE HEINZ-KATO-FURUTA INEQUALITY

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ABSTRACT. The Heinz-Kato-Furuta inequality; if A and B are positive operators on H satisfying $T^*T \le A^2$ and $TT^* \le B^2$ for a given operator T on H, then

$$|\langle T|T|^{\alpha+\beta-1}x, y\rangle| \le \| A^\alpha x \| \| B^\beta y \|$$

for all $\alpha, \beta \in [0,1]$ with $\alpha+\beta \ge 1$ and $x, y \in H$, has several extensions and improvements. On the other hand, the Bernstein inequality for selfadjoint operators is generalized for arbitrary operators. Recently we gave it Bessel type extensions. So we try to give a simultaneous Bessel type extension of the Heinz-Kato-Furuta inequality and the Bernstein inequality. As an application of the Furuta inequality, we obtain Furuta type extension of them. Moreover it is considered under the chaotic order, i.e., $\log A \ge \log B$ for positive invertible operators A and B. Finally we discuss a simultaneous extension of the Heinz-Kato-Furuta inequality and the Selberg inequality.

1. INTRODUCTION.

In what follows, an operator means a bounded linear one acting on a complex Hilbert space H. An operator T is positive, denoted by $T \ge 0$, if $\langle Tx, x\rangle \ge 0$ for all $x \in H$. The order $S \le T$ means that S and T are selfadjoint operators and $S - T$ is positive. Let $T = U|T|$ be the polar decomposition of T on H in the below.

We first cite the Heinz-Kato-Furuta inequality [20], [21] which is shown by a generalized Schwarz inequality via the Löwner-Heinz inequality:

The Heinz-Kato-Furuta inequality. *Let T be an operator on H. If A and B are positive operators on H such that $T^*T \le A^2$ and $TT^* \le B^2$, then*

$$|\langle T|T|^{\alpha+\beta-1}x, y\rangle| \le \| A^\alpha x \| \| B^\beta y \| \tag{1.1}$$

holds for all $\alpha, \beta \in [0,1]$ with $\alpha + \beta \ge 1$ and $x, y \in H$.

In the below, we call it the HKF inequality. We here remark that the Heinz-Kato inequality is just the case $\alpha + \beta = 1$ in above, cf. [4]. In [12, Theorem 2], we proposed the following improvement of the HKF inequality and gave conditions under which the equality holds:

1991 *Mathematics Subject Classification.* Primary 47A30,47A63.

Key words and phrases. Heinz-Kato-Furuta inequality, Bernstein inequality, Selberg inequality, Furuta inequality and chaotic order.