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ANALYTIC CLUSTER SETS

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ABSTRACT. We study the cluster sets for analytic functions in the unit disk. Lindelöf and Meier types theorems are proved for analytic cluster sets.

1. INTRODUCTION

Let $D=\{z:|z|<1\}$ be the unit disk in the finite complex plane $\mathbf{C}$ and $\Gamma=\{z:|z|=1\}$. For each pair of points $a,b=\in D$ the hyperbolic distance between a and b is defined by

$$\sigma(a,b)=\frac{1}{2}\log\frac{|1-\bar{a}b|+|a-b|}{|1-\bar{a}b|-|a-b|}$$

and if L is any curve contained in D, we set

$$\sigma(a,L)=\inf_{b\in L}\sigma(a,b).$$

Let $h(\zeta,\alpha)$ denote the chord which is terminating at the point $\zeta=e^{i\theta}\in\Gamma$ and make up the angle of openning α, $-\frac{\pi}{2}<\alpha<\frac{\pi}{2}$, with the radius of D at ζ. The subset bounded by the chords $h(\zeta,\alpha_1)$ and $h(\zeta,\alpha_2)$ and by the circle $|z-\frac{1}{2}\zeta|=\frac{1}{2}$ is denoted by $\Delta(\zeta,\alpha_1,\alpha_2)$ (or, simply, by $\Delta(\zeta)$ if we are not interested in the magnitude of angle $\Delta(\zeta,\alpha_1,\alpha_2)$).

Let f be an arbitrary real or complex-valued function defined on D. We denote by $C(f,\zeta,D)$, $C(f,\zeta,h(\zeta,\alpha))$ and $C(f,\zeta,\Delta(\zeta))$, respectively, the cluster set of f at the point $\zeta=e^{i\theta}\in\Gamma$ with respect to the disk D, the chord $h(\zeta,\alpha)$ and the angle $\Delta(\zeta)$.

A point $\zeta=e^{i\theta}\in\Gamma$ belongs to the set $K(f)$ if $C(f,\zeta,\Delta_1(\zeta))=C(f,\zeta,\Delta_2(\zeta))$ for any two angles $\Delta_1(\zeta)$ and $\Delta_2(\zeta)$ with the vertix at the point ζ. A point $\zeta=e^{i\theta}\in\Gamma$ belongs to the set $C(f)$ if $\bigcap_\Delta C(f,\zeta,\Delta(\zeta))=C(f,\zeta,D)$ (over all angles $\Delta(\zeta)$). By definition, $C(f)\subset K(f)$.

The structure of cluster sets of meromorphic functions in D was studied by many authors (see e.g. [CL], [G], [GH]). For example, by the strengthens version of Meier's theorem [G], for any meromorphic function f in D the unit circle Γ can be represented as union of disjoint sets of Meier points, precised Plessner points $I^*(f)$, set $P(f)$ and a set E of first Baire category and of type F_σ on Γ. The sets $I^*(f)$ and $P(f)$ are disjoint subsets of the set $I(f)$ of Plessner points for a meromorphic function f in D and a point $\zeta=e^{i\theta}\in\Gamma$ belongs to the set $I(f)$ if $\cap_\Delta C(f,\zeta,\Delta(\zeta))=\Omega$, where Ω denotes the Reimann sphere. Moreover, by definition the sets $I^*(f)$ and $P(f)$ are connected with the concept of a P-sequence, related the property of

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