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Note on Kaplansky's Commutative Rings

Tomohiro TANABE and Ryûki MATSUDA *

Let L be a torsion-free abelian (additive) group, and let S be a subsemigroup of L. Assume that $S \ni 0$. Then S is called a grading monoid (or a g-monoid) ([8]). Many technical terms in multiplicative ideal theories for commutative rings R may be defined analogously for g-monoids S. For example, a non-empty subset I of a g-monoid S is called an ideal of S if $S + I \subset I$. An ideal P of S is called a prime ideal of S, if $P \neq S$ and if $x + y \in P$ (for $x, y \in S$) implies $x \in P$ or $y \in P$. An element x of S is called a unit of S, if $x + y = 0$ for some element $y \in S$. An element x of S is called a prime element of S, if $S + x$ is a prime ideal of S. If every non-unit element of S is expressible as a finite sum of prime elements of S, S is called a unique factorization semigroup (or a UFS). Let x, y be elements of S. We say that x divides y, if $y = x + s$ for some $s \in S$. S is called a Noetherian semigroup, if each ideal I of S can be expressible as $I = \bigcup_{i=1}^{n}(S + a_i)$ for a finite number of elements $a_1, \cdots a_n$ of S. $\cdots$. Many propositions in multiplicative ideal theories for commutative rings R are known to hold for g-monoids S (cf. [1], [2] and [6]). Of course, every technical term for commutative rings R can not be necessarily defined for g-monoids S, and every proposition for R can not be necessarily formulated for S. However, the second author conjectures that almost all propositions in multiplicative ideal theories for R hold for S.

The aim of this paper is to prove propositions in Kaplansky's Commutative Rings ([4]) for g-monoids. We will prove for g-monoids S all the propositions in [4, Ch.1 and Ch.2] that can be formulated for S. We will give consecutive numbers for all of our propositions. The case that the proof of some proposition is straightforward, we will omit it's proof.

If an ideal I is properly contained in S, then I is called a proper ideal of S. If, for a proper ideal M, there are no ideals properly between M and S, then M is called a maximal ideal of S.

Let I be an ideal of a g-monoid S, and $x, x_1, \cdots, x_n \in S$. Then we set $(x_1, \cdots, x_n) = \bigcup_{i=1}^{n}(S + x_i)$ and $(I, x) = I \cup (S + x)$. If $I = (a)$ for some

*Department of Mathematics, Ibaraki University