

# A Characterization of Generalized Saddle Points for Vector-Valued Functions via Scalarization\*

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**Abstract.** In this paper, we propose abstract concepts of saddle points of a vector-valued function  $f$  defined on a product  $X \times Y$  of infinite-dimensional sets  $X$  and  $Y$  in locally convex topological vector spaces. Three notions of the generalized saddle points are considered, and a notions of semi-saddle points, which is also known as “Nash equilibrium points” for a two-person nonzero-sum game in game theory, is defined for a pair of scalarized functions. Various necessary conditions, sufficient conditions and existence conditions are explored for each type of the generalized saddle points. These conditions give a connection between each type of the generalized saddle points and the corresponding type of the semi-saddle points.

**Key words.** vector optimization, scalarization, two-person game, cone saddle points, equilibrium points, convex-concave functions.

## 1. Introduction

Saddle point problems are important in the areas of optimization theory and game theory. As for optimization theory, the main motivation for studying saddle points of scalar-valued functions in the past has been their connection with characterizing solutions to dual problems. Also, as for game theory, the main motivation in the past has been the determination of two-person zero-sum games based on the minimax principle. That is, saddle points of the payoff function for the game are optimal strategy pairs for players. Unfortunately, no vector-valued saddle point problems have been formulated in any application in those areas and the other areas. However, we think that they will eventually lead to worthwhile new developments in mathematics as vector optimization has been explored widely. Therefore, we will generalize a saddle point concept and investigate the generalized saddle points which are called “cone saddle points”.

Since scalarization method is of great importance on characterizing and computing for vector optimization theory, we will adopt the same approach to characterize the generalized saddle points. There have recently appeared many papers connected with scalarization in

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