A class of homogeneous Riemannian manifolds

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1. Introduction

R. L. Bishop and B. O'Neill [1] constructed a wide class of Riemannian manifolds of negative curvature by warped product using convex functions. For two Riemannian manifolds B and F, a warped product is denoted by $B\times_f F$ where f is a positive C^∞ function on B. The purpose of this paper is to prove

THEOREM. *Let (F, g) be a Riemannian manifold of constant curvature $K\leqq 0$. Let E^n be an n-dimensional Euclidean space and let f be a positive C^∞ function on E^n. If either $E^n\times_f F$ is homogeneous (Riemannian) or the Ricci tensor of $E^n\times_f F$ is parallel, then $E^n\times_f F$ is locally symmetric.*

The proof of the last theorem is motivated by [2], in which S. Tanno deals with some related problems.

2. The curvature tensor of $E^n\times_f F$

Let (F, g) be a Riemannian manifold and let E^n be a Euclidean n-space. We consider the product manifold $E^n\times F$. For vector fields A, B, C, etc. on E^n, we denote vector fields $(A, 0)$, $(B, 0)$, $(C, 0)$, etc. on $E^n\times F$ by also A, B, C, etc. Likewise, for vector fields X, Y, etc. on F, we denote vector fields $(0, X)$, $(0, Y)$, etc. on $E^n\times F$ by X, Y, etc.

We denote the inner product of A and B on E^n by $\langle A, B\rangle$. Let f be a positive C^∞-function on E^n. Then the (Riemannian) inner product $\langle\ ,\ \rangle$ for $A+X$ and $B+Y$ on the warped product $E^n\times_f F$ at (a, x) is given by (cf. [1].)

$$\langle A+X, B+Y\rangle_{(a,x)}=\langle A, B\rangle_{(a)}+f^2(a)g_x(X, Y).$$

We extend the function f on E^n to that on $E^n\times_f F$ by $f(a,x)=f(a)$. The Riemannian connections defined by $\langle\ ,\ \rangle$ on E^n and $E^n\times_f E$ are denoted by ∇^o and ∇, respectively. The Riemannian connection defined by g on F is denoted by D. Then we have the identities (cf. Lemma 7.3, [1].)