Positive linear maps of Banach algebras with an involution

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1. Introduction

A linear map $T: A \rightarrow B$ is called a *positive linear map* if $T(A^{+}) \subset B^{+}$, where A and B are complex Banach *-algebras, and, A^{+} and B^{+} are the sets of all finite sums of the form $x^{*}x$ ($x \in A$ or $x \in B$.) In [7], we investigated some properties of positive linear maps of Banach *-algebras. In this paper, we shall also consider some properties of positive linear maps of complex *-Banach algebras with an identity (namely, Banach *-algebras with an isometric involution and an identity of norm one)

Let A be a complex *-Banach algebra with an identity e_A. By $\|x\|$, we denote the norm of $x \in A$. Moreover, we denote the well known pseud-norms on A as follows:

$$\|x\|_{1, A} = \sup \{|f(x)|; f \text{ is positive linear functional on } A \text{ such that } f(e_A) \leqq 1\},$$
$$\|x\|_{2, A} = \sup \{(f(x^{*}x))^{\frac{1}{2}}; f \text{ is positive linear functional on } A \text{ such that } f(e_A) \leqq 1\}.$$

Then we have $\|x\|_{1, A} \leqq \|x\|_{2, A} \leqq \|x\|$. If A is a C*-algebra, we have $\|x\|_{1, A} = \|x\|_{2, A} = \|x\|$ for every hermitian element x of A. Moreover $\{x \in A; \|x\|_{1, A} = 0\}$ and $\{x \in A; \|x\|_{2, A} = 0\}$ coincide with the *-radical $R^{(*)}{}_A$ of A. We recall that, if A has an identity, any positive linear map is self-adjoint (namely, $T(x^{*}) = (T(x))^{*}$). The notations given in [7] will be quoted without notice.

2. Operator norm of positive linear map

In [7], we discussed the continuity of positive linear maps of Banach *-algebras. In this section, we consider the operator norm of positive linear map of *-Banach algebras with an identity.

We need the following definition.

DEFINITION 2. 1. *Let A and B be a *-Banach algebra and a C*-algebra respectively, and T be a positive linear map of A into B. Then T is said to satisfy the stronger form of generalized Schwarz inequality provided $T(x^{*})\, T(x) \leqq \|T\|\, T(x^{*}x)$ for every $x \in A$.*

If $T(x)$ is of the form $V^{*}\rho(x)V$ for every $x \in A$, where ρ is a *-representation of A on a complex Hilbert space K, and H is a complex Hilbert space on which B acts, and V is a