On 3-dimensional Riemannian manifolds satisfying a certain condition on the curvature tensor

By

Hitoshi TAKAGI and Kouei SEKIGAWA

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1. Introduction

If a Riemannian manifold M is locally symmetric, then its curvature tensor R satisfies

(*) $$R(X, Y)\cdot R=0 \quad \text{for all tangent vectors } X \text{ and } Y,$$

where the endomorphism $R(X, Y)$ operates on R as a derivation of tensor algebra at each point of M.

Conversely, does this algebraic condition (*) on the curvature tensor field R imply that M is locally symmetric (i. e. $\nabla R=0$) ?

One must exclude the 2-dimentional case, as was already observed by E. Cartan, 1. K. Nomizu has conjectured that the answer is affirmative in the case where M is irreducible and complete and dim. $M\geqq 3$. There are some partial or related results in this direction.

The main purpose of the present paper is to deal with the same problem about 3-dimensional Riemannian manifolds.

2. Reduction of condition (*) and some results

Let M be a 3-dimensional connected Riemannian manifold, then it is well known that the curvature tensor R of M is written in the form

(2. 1) $$R(X, Y)=AX\wedge Y+X\wedge AY-\frac{1}{2}(\text{trace } A)X\wedge Y$$

where A is a field of symmetric endomorphism which corresponds to the Ricci tensor field S, that is, $g(AX, Y)=S(X, Y)$, g being the Riemannian metric and $X\wedge Y$ denotes the endomorphism which maps Z upon $g(Z, Y)X-g(Z, X)Y$.

At a point $x\in M$, let $\{e_1, e_2, e_3\}$ be an orthogonal basis of the tangent space $T_x(M)$ such that $Ae_i=\lambda_i e_i$, $i=1, 2, 3$.

Then, the equation (2. 1) implies