On extreme points of the unit ball of a non-commutative L^p-space with $0<p\leq 1$

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1. Introduction

The notion of an extreme point is playing an important role in the theory of topological vector spaces, in particular, that of Banach spaces. Some properties of a linear map can be stated in terms of extreme points of a certain convex set of linear maps. For example, a non-zero representation of a C^*-algebra is irreducible if and only if it is spatially equivalent to a GNS-representation associated with a pure state (that is an extreme point of the state space). R. Kadison gave a characterization of extreme points of the unit ball of a C^*-algebra, and he applied it to classify isometries between C^*-algebras. Thus it is fundamental to characterize extreme points of the unit ball of a (quasi-) Banach space associated with an operator algebra. For a commutative L^p-space $L^p(X, \mu)$ of measurable functions, the results are well-known. In case of $1<p<\infty$, the unit sphere is precisely the set of all extreme points of the unit ball. If $p=\infty$, the set of all extreme points is exactly the unitary group of $L^\infty(X, \mu)$ ([**6**, Chapter I, Lemma 10.11]). For $0<p\leq 1$, there exists an extreme point if and only if the measure space (X, μ) has an atom.

Our first aim in this paper is to give a necessary and sufficient condition for the existence of an extreme point of the closed unit ball of a non-commutative L^p-space. Secondarily, we shall determine the form of each extreme point completely. However, when $1<p<\infty$, it is shown that the Clarkson-McCarthy's inequality holds for non-commutative L^p-spaces associated with von Neumann algebras (see [**1**], [**3**], [**7**]). Therefore they are uniformly convex, in particular, strictly convex. Thus the set of all extreme points coincides with the unit sphere. For $p=\infty$, L^∞-space was defined to be the von Neumann algebra itself and so it is a C^*-algebra. If S is the unit ball of a C^*-algebra A, the following facts are well-known; (i) there exists an extreme point x in S if and only if A is unital; and (ii) when A is unital, then $x\in S$ is extreme if and only if x^*x is a projection such that $(1-x^*x)A(1-xx^*)=\{0\}$ ([**6**, Chapter I, Theorem 10.2]). Therefore, we may concentrate our attention to the case of $0<p\leq 1$. Finally we also consider