A Theorem of Schur Type for Locally Symmetric Spaces

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Abstract. By showing hidden hypotheses in Schur's lemma on spaces of constant curvature we get a new version for locally symmetric spaces.

0. Statement

Let M be a connected Riemannian manifold with dimension $n\geq 3$. Schur proved in 1886 that M is a space of constant curvature if the sectional curvature depends only on the points (see [2], [3]). In the present note we improve the theorem and have a theorem of the same type for locally symmetric spaces.

Let ∇ be the *Riemannian connection* and let R be the *Riemannian curvature tensor* given by

$$\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z=R(X, Y)Z$$

where X, Y, Z are vector fields and $[\cdot, \cdot]$ is the Lie bracket. We say that the eigenspaces of R are *parallel* if the following condition is satisfied: For any geodesic ν and for any unit parallel vector field v along ν the eigenspaces of $R(\cdot, v)v: T_pM \longrightarrow T_pM$ are parallel along ν where p is the foot point of v. The locally symmetric spaces have this property. If the sectional curvature depends only on the points, then the condition is automatically satisfied, since the eigenspaces of $R(\cdot, v)v: T_pM \longrightarrow T_pM$ is either $v^\perp$ or T_pM which are parallel along ν, where $v^\perp$ is the space orthogonal to v.

Theorem. *Let M be a connected Riemannian manifold with dimension $n\geq 3$. Suppose there exist functions $c_1, \dots, c_i$ on M such that (1) the distinct eigenvalues of $R(\cdot, v)v: T_pM \longrightarrow T_pM$ are $c_1(p), \dots, c_i(p)$ for any point $p\in M$ and any unit vector $v\in T_pM$ with $c_i(p)=0$ and (2) if $c_j=\lambda_j c_1$ then λ_j are constants on M for $j=1, \dots, i-1$ (always $\lambda_1=1$* and *$\lambda_i=0$). If the eigenspaces of R are parallel and dim Ker $R(\cdot, v)v\leq n-2$ for any unit vector v, then M is a locally symmetric space.*

Here, $Ker\ R(\cdot, v)v$ is by definition the kernel of $R(\cdot, v)v$: $T_pM \longrightarrow T_pM$. The