Fixed Points of Expanding Maps

By
Tetsuo KANEKO and Kazu-Hiro SAKAI

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1. Introduction

Let $\{f_i\}_{i=1}^{\infty}$ be a convergent sequence of maps from a space X into itself and let f_0 be a limit map. When does there exist a sequence of fixed points a_i of f_i such that $\{a_i\}_{i=1}^{\infty}$ converges to a_0 for each fixed point a_0 of f_0. In [3] Rosen proved that it holds when X is a compact connected ANR and f_i is an open ϵ-locally expansive map for $i=0$, 1, 2, ..., and $\{f_i\}_{i=1}^{\infty}$ converges uniformly to f_0. In [2] Hu and Rosen recently showed that for a compact connected locally connected metric space, the ANR requirement can be dropped.

In this paper we show that in the hypothesis of the Theorem 4.8 in [2], if $\{f_i\}_{i=1}^{\infty}$ is a sequence of expanding maps with common ε and λ, the uniform convergence may be replaced by pointwise convergence and f_0 may be any map with a fixed point.

2. Definition and lemmas

Let (X, d) be a compact metric space. A continuous map $f: X\to X$ is called an ε-local expansion if there are $\varepsilon>0$ and skewness $\lambda>1$ such that $0<d(x, y)<\varepsilon$ implies $d(f(x), f(y))>\lambda d(x, y)$.

We call a continuous map f to be expanding if f is open and ε-local expansion for some $\varepsilon>0$ and $\lambda>1$.

Rosenholtz showed in [4] that if X is a compact connected metric space, such map f has a fixed point.

LEMMA 1. *If X is a compact connected locally connected space with metric d and if $\{f_i\}_{i=1}^{\infty}$ is a sequence of expanding maps of X onto itself with common ε and λ, then there is $\delta_0>0$ $(\delta_0<\varepsilon)$ such that x, $y\in X$ with $d(f_i(x), y)<\delta_0$ implies $B_{\delta_0/\lambda}(x)\cap f_i^{-1}(y)\neq\phi$ for $i=1$, 2, 3, ..., where $B_{\alpha}(x)=\{y\in X: d(x, y)<\alpha\}$.*

PROOF. According to Lemma 2 in [3], there is a finite open cover $\{W_\beta\}$ of X such that for each β and for $i=1, 2, 3, \ldots$, W_β is connected and diam $W_\beta<\varepsilon$ and f_i maps every component of $f_i^{-1}(W_\beta)$ homeomorphically onto W_β and furthermore evey component C of $f_i^{-1}(W_\beta)$ has diameter $<\varepsilon$. Let $\delta_0>0$ $(\delta_0<\varepsilon)$ be a Lebesgue number for $\{W_\beta\}$. If x, $y\in X$ and $d(f_i(x), y)<\delta_0$, then there is some W_β containing $f_i(x)$ and y. Let C be the