On some cases where n-positivity coincides with complete positivity*

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The difference between the notions of positivity and complete positivity of various types of maps associated to operator algebras is discussed in [6]. It is shown that the commutativity of one of the associated algebras makes no difference between these two notions and conversely when this is the case, it implies the commutativity of one of the associated algebras. In this paper we investigate some typical cases where the notion of n-positivity coincides with that of complete positivity for those maps not necessarily defined between C^*-algebras. The case where maps are defined between C^*-algebras has been treated in Choi [1]. Let E and F be C^*-algebras, the duals of C^*-algebras, or the preduals of von Neumann algebras. Then our result states that every n-positive linear map of the $n\times n$ matrix space $M_n(E)$ to the space F is completely positive provided the associated algebra of E is abelian and is also the case for a n-positive linear map of E to $M_n(F)$ provided the associated algebra of F is abelian.

Let A be a C^*-algebra and M a von Neumann algebra. We can equip the $n\times n$ matrix space over the dual space A^*,

$$M_n(A^*)=\{f=[f_{ij}]\,|\,f_{ij}\in A^*\},$$

with the order of the dual space of the matrix C^*-algebra $M_n(A)=\{a=[a_{ij}]\,|\,a_{ij}\in A\}$. Let M_* be the predual of a von Neumann algebra M, then we can also equip the space $M_n(M_*)$ with the relative order in $M_n(M^*)$. Let E and F be C^*-algebras, the duals of C^*-algebras, or the preduals of von Neumann algebras. A linear map τ of E to F is said to be n-positive if the map

$$\tau_n : [c_{ij}]\in M_n(E)\longmapsto[\tau(c_{ij})]\in M_n(F)$$

is positive and τ is said to be completely positive if τ is n-positive for every positive integer n.

The following lemma is an immediate consequence of [2, Lemmas 4. 1 and 4. 3].

LEMMA 1. *Let E be a C^*-algebra, the dual of a C^*-algebra, or the predual of a von Neumann algebra. Then every n-positive linear map of the $n\times n$ complex matrix algebra $M_n(\boldsymbol{C})$ to E is completely positive and moreover every n-positive linear map of E to $M_n(\boldsymbol{C})$ is completely positive.*

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