A note on the cut loci and the first conjugate loci of riemannian manifolds

By

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1. Introduction

Let M be an n-dimensional complete riemannian manifold and $d(M)$ (resp. K_M) its diameter (resp. sectional curvature), C_p (resp. Q_p^1) the cut locus (resp. the first conjugate locus) in the tangent space M_p to M at $p \in M$. It is well known that $K_M \geq \delta > 0$ implies $\pi/\sqrt{\delta} \geq d(M)$ (Myers [1]). On the other hand, the following is a classical problem concerning the infimum of the diameter $d(M)$ of a riemannian manifold M.

PROBLEM 1. Does $\delta \geq K_M$ imply $d(M) \geq \pi/\sqrt{\delta}$ for a simply connected riemannian manifold?

In this problem, we can not remove the assumption of simple connectivity for M. In fact, an n-dimensional real projective space $\boldsymbol{RP}^n$ with the canonical metric of constant curvature δ has the diameter $\pi/2\sqrt{\delta}$. Furthermore, if M is non-compact, then the diameter of M is infinite by Hopf-Rinow's theorem. Therefore, in the sequel, we may assume that M is compact. Now it may be easily understood that if the answer to the following problem 2 is affirmative, we have the same answer to the above problem 1 by virtue of Morse-Schoenberg's theorem.

PROBLEM 2. (Weinstein [2]). Is there a point $p \in M$ such that $C_p \cap Q_p^1 \neq \emptyset$ for a compact simply connected riemannian manifold M?

In this paper, we give some partial answers to problem 2. In §2, we introduce the notion of flaps of a riemannian manifold and prove proposition 1 by making use of the properties of flaps. Here "flap" intuitively means the pasting part in making a cylinder from a rectangle. In §3 of this note, the following theorem is proved.

THEOREM. *Let M be a compact simply connected riemannian manifold with a point $p \in M$ satisfying the condition that if $\Omega(p, q)$ is non-degenerate, there is at most one geodesic of index 1 in $\Omega(p, q)$. Then C_p and Q_p^1 have an intersection.*

This statement includes a result of Warner [3] that if there are no geodesics of index 1 in a simply connected manifold M, the cut locus and first conjugate locus coincide.

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