On the structure of p-class groups of certain number fields II

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1. Introduction

Let p be a rational odd prime and let k be an algebraic number field of finite degree, whose class number h_k is prime to p. Let K/k be a cyclic extension of degree p, let $\mathfrak{p}_1$, ……, $\mathfrak{p}_t$ be the prime ideals of k, ramified in K, and assume $\mathfrak{p}_1, \cdots\cdots, \mathfrak{p}_t$ are prime to p. If $\#(I(\mathfrak{p}_i)/H(\mathfrak{p}_i)=p$ for $i=1, \cdots\cdots, t$, then we can study the p-class group M_K of K analogously to the case $k=\boldsymbol{Q}$, where $I(\mathfrak{p}_i)$ denotes the ideal group of k, prime to $\mathfrak{p}_i$ $P_{\mathfrak{p}_i}$, the ray mod $\mathfrak{p}_i$ and $H(\mathfrak{p}_i)=I(\mathfrak{p}_i)^pP_{\mathfrak{p}_i}$. From Lemma 1 it follows that if k does not contain the primitive p-th roots of unity, then there are infinitely many such $\mathfrak{p}_i$'s which satisfy some conditions each other.

In the present paper we treat the existence of cyclic extensions K/k's of degree p and t-tuples of prime ideals $\mathfrak{p}_1, \cdots\cdots, \mathfrak{p}_t$, which have some properties. Unless otherwise stated the notation of [4] will be taken over. In particular $\mathfrak{o}$ denotes the maximal order of the cyclotomic field of p-th roots of unity and $\mathfrak{p}$ denotes the prime divisor of p in $\mathfrak{o}$. Let K/k be a cyclic extension of degree p, in which only $\mathfrak{p}_1, \cdots\cdots, \mathfrak{p}_t$ are ramified. Then for $\mathfrak{p}_1, \cdots\cdots, \mathfrak{p}_t$ the structue of p-class group M_K, in general, is not determined uniquely. In fact we can prove the following theorem.

THEOREM 1. *Let k be an algebaic number field of finite degree such that $p \nmid h_k$ and $k \ni \xi_p$, where ξ_p denotes a primitive p-th root of unity. Then for any given natural number t ($\geqq 3$), there exist infinitely many t-tuples of prime ideals $\mathfrak{p}_1, \cdots\cdots, \mathfrak{p}_t$ of k, which satisfy the following conditions:*

there are cyclic extensions K'/k and K''/k in which only $\mathfrak{p}_1, \cdots\cdots, \mathfrak{p}_t$ are ramified, such that rank $M_{K'}=t-1$ and rank $M_{K''}\geqq 2t-3-u$, where u denotes the p-rank of unit group E_k of k.

Let $\mathfrak{p}_1, \cdots\cdots, \mathfrak{p}_t$ be prime ideals of k such that $\#(I(\mathfrak{p}_i)/H(\mathfrak{p}_i))=p$ for $i=1, \cdots\cdots, t$, let K/k be a cyclic extension of degree p, in which only $\mathfrak{p}_1, \cdots\cdots, \mathfrak{p}_t$ are ramified and let L be the p-genus field (i.e. p-part of the genus field) with respect to K/k. In the case $k=\boldsymbol{Q}$,

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