On strong consistency of a sequential estimator of probability density

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(Received October 30, 1977)

1. Introduction and Summary

Let $F(x)$ be a probability distribution function on the real line. It is well known that assuming that the singular part is identically zero, $F(x)$ can be uniquely decomposed into

(1. 1) $$F(x)=F_1(x)+F_2(x),$$

where $F_1(x)$ is an absolutely continuous function and $F_2(x)$ is a pure step function with steps of magnitude, say, S_i at the points $x=x_i$, $i=0, \pm 1, \pm 2, \cdots\cdots$ and finally both $F_1(x)$ and $F_2(x)$ are non-decreasing. If the singular part is identically zero as has been assumed here, $F_1(x)$ has a density function $f(x)$ almost everywhere, namely,

(1. 2) $$dF_1(x)/dx=f(x) \quad \text{a.e.x.}$$

At a point of continuity x_0' of $F(x)$ its density is clearly $f(x_0')$.

Let X_1, X_2, X_3, $\cdots\cdots$ be a sequence of independent identically distributed random variables with the common distribution function $F(x)$. We shall consider the problem of estimating the density $f(x)$ at all points of continuity of $F(x)$ and also of $f(x)$ as has been seen in (1. 2) from X_1, X_2, X_3, $\cdots\cdots$. The kernel estimate of f from X_1, X_2, X_3 $\cdots\cdots$, X_n is given by

(1. 3) $$f_n(x)=(B_n/n)\sum_{j=1}^{n} K(B_n(X_j-x)),$$

where K, the kernel, is an arbitrary bounded probability density on the real line and $\{B_n\}$ is a sequence of positive numbers. For some conditions on K and $\{B_n\}$, Murthy [1] proved that $f_n(x_0')$ is a consistent estimate of $f(x_0')$ at a point of continuity x_0' of the distribution $F(x)$ and also of the density $f(x)$ under the condition

(1. 4) $$\sum_i S_i/|x_0'-x_i|<\infty.$$

(That is $f_n(x_0')\longrightarrow f(x_0')$ in prob. as $n\rightarrow\infty$).

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