A note on invariant measures for the Galton-Watson process with state-dependent immigration

By

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1. Introduction

Consider the Galton-Watson branching process with state-dependent immigration, where immigration is allowed in a generation iff the previous generation was empty (Pakes (1971) [3]).

Let $A(x)=\sum_{j=0}^{\infty} a_j x^j$ and $B(x)=\sum_{j=0}^{\infty} b_j x^j$ $(|x|\leqq 1)$ be the probability generating functions of the offspring and immigration distributions respectively. We shall assume that

1) $\quad 0<a_0, \quad a_0+a_1, \quad b_0<1$, and

2) $\quad \alpha=A'(1-)<\infty$.

Denote the size of the n-th generation by X_n $(n=0, 1, \cdots)$.

Now we discuss the problem of the existence and uniqueness of invariant measure of the Markov chain $\{X_n\}$, that is, a non-negative sequence $\{\mu_i\}$ $(i=0, 1, \cdots; \mu_i>0$ for some $i)$ such that

$$\mu_j=\sum \mu_i p_{ij} \qquad (j=0, 1, \cdots),$$

where p_{ij} is the one-step transition probability from state i to j.

The following results are given by Pakes (1971) [3].

LEMMA A. *Suppose an invariant measure, $\{\mu_i\}$, of the Markov chain $\{X_n\}$* exists. Then *$U(x)=\sum_{i=0}^{\infty} \mu_i x^i$ converges for $x\in[0, q)$ and satisfies the functional equation*

$$U[A(x)]=U(x)+\mu_0(1-B(x)), \qquad \mu_0>0 \tag{1}$$

for $x\in[0, q)$, where q is the least positive solution of $x=A(x)$, so that $q=1$ if $\alpha\leqq 1$ and *$0<q<1$ if $\alpha>1$.*

THEOREM B. *When $\alpha\leqq 1$, the Markov chain, $\{X_n\}$, possesses a unique (up to a constant multiplier) invariant measure. And we obtain*