A generalization of Malliavin's tauberian theorem

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1. Introduction

In this paper we generalize the theorem by P. Malliavin together with the formula of Å. Pleijel to deduce the behaviour of a positive measure μ defined on a sector of the complex plane:

$$\Gamma=\{\lambda : |\arg\lambda|\leqq\theta\},\quad 0\leqq\theta<\pi/4$$

from the behaviour of a transform

$$f(z)=\int_{\Gamma}\frac{\mu(d\lambda)}{\lambda-z},\quad z\notin\Gamma.$$

P. Malliavin [1] proved the following

THEOREM A. *Let $\sigma(\lambda)$ be a non-decreasing function for $\lambda\geqq 0$. Suppose that*

$$\int_{0}^{\infty}\frac{d\sigma(\lambda)}{\lambda-z}=a(-z)^{-\alpha}+O(|z|^{-\beta})$$

as $z\rightarrow\infty$ with $|\operatorname{Im} z|=|z|$ and $\operatorname{Re} z\geqq 0$, where $0<\alpha<\beta<1$, $0<\gamma<1$, $a>0$.

Then as $X\rightarrow\infty$

$$\sigma(X)=a\frac{\sin(1-\alpha)\pi}{(1-\alpha)\pi}X^{1-\alpha}+O(X^{\gamma-\alpha})+O(X^{1-\beta}).$$

Å. Pleijel [2] proved this theorem in a very elementary way. His proof uses the following approximate inversion formula for the Stieltjes transform of a positive measure:

THEOREM B. *Let $\sigma(\lambda)$ be a non-decreasing function for $\lambda\geqq 0$ and put*

$$f(z)=\int_{0}^{\infty}\frac{d\sigma(\lambda)}{\lambda-z},\quad z\notin(0,\infty).$$

Then for $X,\ Y>0$

$$\left|\sigma(X)-\sigma(0)-\frac{1}{2\pi i}\int_{L(Z)}f(z)dz+\frac{Y}{\pi}\operatorname{Re} f(Z)\right|\leqq Y\operatorname{Im} f(Z),$$