The epuivalence of two definitions of homotopy sets for Kan complexes

By

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As we remarked in §§ 1 and 2 of [1], the following proposition holds. The purpose of this paper is to give its proof. Free use will be made of the definitions and notations of [1].

PROPOSITION 1. 1°. *If* (K, L) *is a Kan pair with base point* $\varphi\in L_0$, DEFINITIONS 1.7 *and* 1.10 *in* [1] *of* $\pi_n(K, L, \varphi)$ *are equivalent for* $n\geqq 0$. 2°. *If* $(K; L, M)$ *is a Kan triad with base point* $\varphi\in(L\cap M)_0$, DEFINITIONS 2.4 *and* 2.7 *in* [1] *of* $\pi_n(K; L, M, \varphi)$ *are equivalent for* $n\geqq 2$. *I.e. the natural embedding map* $i_k: K\to S|K|$ *given in* [7] *induces one-to-one onto maps* $(i_k)_*: \pi_n(K, L, \varphi)\to\pi_n(S|K|, S|L|, i_k(\varphi))$ *and* $(i_k)_*: \pi_n(K; L, M, \varphi)\to\pi_n(S|K|; S|L|, S|M|, i_k(\varphi))$ *where* π_n *means the set defined by* DEFINITIONS 1.7 *and* 2.4 *in* [1].

Proof of 1°. The equivalence follows from THEOREM 7.3 in [1], REMARK 1 in [3, §4] and the five lemma for $n\geqq 2$, and by their definitions for $n=0$.

To show that $(i_k)_*$ is one-to-one onto for $n=1$, consider $\pi_1(K, L, \varphi)$ and $\pi_1(S|K|, S|L|, i_k(\varphi))$. In this case we may assume that K is connected, i.e. $\pi_0(K, \varphi)=0$. Then we can construct the c. s. s. group $G(K;\varphi)$ which is a loop complex of K rel. φ [2, THEOREM 9.2]. Put $U=G(K;\varphi)\times_t K$, $C=G(K;\varphi)\times_t L$ and $\psi=(e_0, \varphi)\in U_0$ where t is a twisting function defined by $t\sigma=\bar{\sigma}$, e_0 is the identity element of the group $G(K;\varphi)_0$. By LEMMA 9.3 in [2] U is contractible. Let $p: U\to K$ be given by $p(\rho, \sigma)=\sigma$ for $(\rho, \sigma)\in U$. Then p is a fibre map: $(U, C, \psi)\to(K, L, \varphi)$ and (U, C) is a Kan pair. By THEOREM 8.3-2) and PROPOSITION 8.2 in [1], $p_*: \pi_1(U, C, \psi)\to\pi_1(K, L, \varphi)$ and $(S|P|)_*: \pi_1(S|U|, S|C|, i_U(\psi))\to\pi_1(S|K|, S|L|, i_k(\varphi))$ are one-to-one onto.

Consider the following commutative diagram:

$$\begin{array}{ccccc} \pi_1(K, L, \varphi) & \xleftarrow{\;p_*\;} & \pi_1(U, C, \psi) & \xrightarrow{\;\delta\;} & \pi_0(C, \psi) \\ \downarrow (i_k)_* & & \downarrow (i_U)_* & & \downarrow (i_C)_* \\ \pi_1(S|K|, S|L|, i_k(\varphi)) & \xleftarrow{\;(S|p|)_*\;} & \pi_1(S|U|, S|C|, i_U(\psi)) & \xrightarrow{\;\delta'\;} & \pi_0(S|C|, i_C(\psi)), \end{array}$$

where δ and δ' are the boundary operations induced by the 0-th face operation, $(i_C)_*$ is one-to-one onto [3, §4 REMARK 1]. Therefore to show that $(i_k)_*$ is one-to-one