Homotopy theory of c.s.s. pairs and triads

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Our main purpose is to give the definitions of the homotopy groups of a c.s.s. pair (or triad) and of homotopy between maps of one of c.s.s. pair (or triad) into another after the combinatorial manner as [5] [6]. And we study the fundamental properties of these notions, for example, those properties mentioned by S. T. Hu [2] as the axioms of homotopy theory of topological space, or the exactness of the lower and upper homotopy sequences of c.s.s. triad. All of these properties may be verified combinatorially.

§ 1. The homotopy gruops of c.s.s. pairs.

In this note, K_n means the collection of all n-simplices of c.s.s. complex K, $\sigma\varepsilon^i$ and $\sigma\eta^i$ mean the *i-th face* and the *i-th degeneracy* of simplex σ.

Definition 1.1. For $\sigma_i \in K_n$, symbol

$$\overset{(0)}{[\sigma_0}, \cdots, \overset{(l-1)}{\sigma_{l-1}}, \overset{(l)}{\square}, \overset{(l+1)}{\sigma_{l+1}}, \cdots, \overset{(n+1)}{\sigma_{n+1}]}$$

is called **an** *equation in K*, and means that $\sigma_i\varepsilon^{j-1}=\sigma_j\varepsilon^i$, $0\leqq i<j\leqq n+1$, $i, j\neq l$ (*match condition*). If there exists $\sigma\in K_{n+1}$ such that $\sigma\varepsilon^i=\sigma_i$, $0\leqq i\leqq n+1$, $i\neq l$, σ and $\sigma\varepsilon^l$ are called *solvent* and *solution* of this equation respectively.

If each equation in K has at least one solvent, K is called a *Kan complex.* (This is a complex which satisfies the *extension condition* [3] [4]).

D. M. Kan [5] gave the following definition:

Definition 1.2. Two simplices σ and τ of K_n $(n\geqq 0)$ is called *homotopic* (notation $\sigma\sim\tau$ or $\rho:\sigma\sim\tau$) if

(a) their faces coincide, i.e. $\sigma\varepsilon^i=\tau\varepsilon^i$ for all i

(b) there exists $\rho\in K_{n+1}$ such that $\rho\varepsilon^n=\sigma$, $\rho\varepsilon^{n+1}=\tau$ and $\rho\varepsilon^i=\sigma\varepsilon^i\eta^{n-1}=\tau\varepsilon^i\eta^{n-1}$, $0\leqq i\leqq n-1$.

We have

Proposition 1.3. *Let K be a Kan complex. Two n-simplices σ and τ of K are homotopic if and only if*

(a) $\sigma\varepsilon^i=\tau\varepsilon^i$ *for all i*