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REVERSES OF OPERATOR INEQUALITIES ON OPERATOR MEANS

MASATOSHI FUJII*, RITSUO NAKAMOTO** AND SATIYO SUGIYAMA*

ABSTRACT. In this note, we improve the non-commutative Kantorovich inequality as follows: If A, B satisfy $0 < m \le A, B \le M$, then for each $\mu \in [0,1]$

$$A \nabla_\mu B \le \frac{M \nabla_\mu m}{M \,!_\mu\, m} A \,!_\mu\, B,$$

where $A \,!_\mu\, B$ is the μ-harmonic mean and $A \nabla_\mu B$ is the μ-arithmetic mean. Next we discuss the optimality of the constant $(\sqrt{M} - \sqrt{m})^2$ in the difference reverse inequality

$$A \nabla B - A \,!\, B \le (\sqrt{M} - \sqrt{m})^2$$

for all positive invertible A, B with $0 < m \le A, B \le M$.

In addition, we compare the μ-geometric mean $A \,\sharp_\mu\, B$ with $A \nabla_\mu B$, $A \,!_\mu\, B$ and $\frac{1}{2}(A \nabla_\mu B + A \,!_\mu\, B)$ for positive operators A and B.

1. Noncommutative Kantorovich inequality. Let Φ be a unital positive linear map on $B(H)$, the C^*-algebra of all bounded linear operators on a Hilbert space H. Then Kadison's Schwarz inequality asserts

$$\Phi(A^{-1})^{-1} \le \Phi(A) \tag{1}$$

for all positive invertible $A \in B(H)$.

If Φ is defined on $B(H) \oplus B(H)$ by

$$\Phi(A \oplus B) = \frac{1}{2}(A + B) \quad \text{for } A, B \in B(H), \tag{2}$$

then Φ satisfies

$$\Phi((A \oplus B)^{-1})^{-1} = A \,!\, B, \quad \Phi(A \oplus B) = A \nabla B \tag{3}$$

for all positive invertible $A, B \in B(H)$, where $A \,!\, B$ is the harmonic operator mean and $A \nabla B$ is the arithmetic operator mean in the sense of Kubo-Ando [5]. Consequently, Kadison's Schwarz inequality implies the arithmetic-harmonic mean inequality, i.e., $A \,!\, B \le A \nabla B$, cf. [1] and [3].

By the same discussion as in above, the weighted arithmetic-harmonic mean inequality, i.e., $A \,!_\mu\, B \le A \nabla_\mu B$ for $\mu \in [0,1]$, is proved.

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