Nihonkai Math. J.
Vol.17(2006), 77-88

Groups, Lie algebras and Gauss decompositions for one dimensional tilings

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Abstract

For one dimensional tilings, we will define associated groups and Lie algebras. Then, we will prove that the groups have Gauss decompositions as well as that the Lie algebras also have additive Gauss decompositions.

Key words: tiling, group, Lie algebra, Gauss decomposition

0. Introduction. In this paper, we will give a certain Lie theoretical approach to tilings, and establish some basic decompositions. What is a tiling ? A tiling is roughly a decomposition or a filling of a given space using suitable pieces. Recently it was found that lots of mathematical areas are deeply related to tilings and associated topics. However, not so many algebraic approaches have been given. This article shows that it is possible to study tilings using Lie theory. Mathematically it is very nice to have certain pure algebraic objects (invariants) arising from tilings.

$$\boxed{\begin{matrix}\text{Tiling}\\ \text{Structure}\end{matrix}} \begin{matrix}\longrightarrow\\ \longleftarrow\end{matrix} \boxed{\begin{matrix}\text{Mathematics}\\ \text{Pure Algebra}\end{matrix}}$$

Here we will discuss a tiling of the real line $\mathbb{R}$. Then, we will construct tiling monoids (Section 2), tiling bialgebras (Section 3), tiling Lie algebras (Section 4) and tiling groups (Section 5). Then we will establish Gauss decompositions for our tiling groups (Sections 6,7,8). Also we will reach certain additive Gauss decompositions for tiling Lie algebras (Section 9). Those decompositions are fundamental in algebra, which can be very helpful to study many kinds of invariants for mathematical objects.

Next, we will make a rough review of Gauss decompositions. Let us consider the following system of linear equations:

$$\begin{cases} ax + by = s \\ cx + dy = t \end{cases}$$

Generically we can solve it using the so-called Gauss eliminaton. This is corresponding to the following group theoretical decomposition over the field $\mathbb{C}$ of complex numbers (cf. [12]):

$$GL_2(\mathbb{C}) = \begin{pmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \mathbb{C} & 1 \end{pmatrix} \begin{pmatrix} \mathbb{C}^\times & 0 \\ 0 & \mathbb{C}^\times \end{pmatrix} \begin{pmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{pmatrix}$$