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AN ESTIMATION OF QUASI-ARITHMETIC MEAN BY ARITHMETIC MEAN AND ITS APPLICATIONS

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ABSTRACT. The quasi-arithmetic mean inequality says that if f is an increasing strictly convex function on an interval I, then $f^{-1}(\langle f(A)x,x\rangle) \geq \langle Ax,x\rangle$ for all unit vectors x in a Hilbert space H and a selfadjoint operator A on H, whose spectrum is contained in I. In this paper, we consider reverse inequalities of the quasi-arithmetic mean inequality. For each $\lambda > 0$ we observe an upper bound of a difference

$$f^{-1}(\langle f(A)x,x\rangle) - \lambda\langle Ax,x\rangle.$$

We find a condition on vectors x which attain the optimal bounds.

Replacing a given function $f(t)$ by a power, the logarithmic and the exponential function, we show these reverse quasi-arithmetic mean inequalities and equality conditions, in which the obtained constants are expressed by a generalized Kantorovich constant, the Specht ratio and the logarithmic mean.

1. INTRODUCTION

Let f be a strictly increasing continuous function on an interval I. Then

$$f^{-1}\left(\frac{1}{n}\sum_{i=1}^{n} f(a_i)\right) \tag{1.1}$$

is called the quasi-arithmetic mean of $a = (a_1, \ldots, a_n) \in I^n(\subset \mathbb{R}^n)$ by f (cf. [15]). Typical examples are arithmetic, geometric and harmonic means which correspond to functions $f(t) = t$, $\log t$ and $-\frac{1}{t}$, respectively.

Throughout this paper, an operator means a bounded linear operator on a Hilbert space H. For each unit vector $x \in H$, we consider

$$f^{-1}(\langle f(A)x,x\rangle) \tag{1.2}$$

for all selfadjoint operators A whose spectra are contained in I, as an operator version of the quasi-arithmetic mean (1.1). Incidentally, $\langle Ax,x\rangle$ is regarded as the arithmetic mean. Indeed, (1.1) is obtained by putting $A = \mathrm{diag}(a_1, \ldots, a_n)$ and $x = \frac{1}{\sqrt{n}}\begin{pmatrix}1\\ \vdots \\ 1\end{pmatrix}$ in (1.2), and obviously $\langle Ax,x\rangle = \frac{1}{n}\sum a_i$. If we choose the logarithmic function $f(t) = \log t$, then its quasi-arithmetic mean $\exp\langle(\log A)x,x\rangle$ for a fixed unit

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