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ABSTRACT. For two bounded positive linear operators $a, b$ on a Hilbert space, we give conditions which imply the commutativity of $a, b$. Some of them are related to well-known formulas for indefinite elements, e.g., $(a + b)^n = \sum_k \binom{n}{k} a^{n-k} b^k$ etc. and others are related to the property of operator monotone functions. We also give a condition which implies the commutativity of a C*-algebra.

1. INTRODUCTION

Ji and Tomiyama ([5]) give a characterization of commutativity of C*-algebra, where they also give a condition that two positive operators commute. For bounded linear operators on a Hilbert space $\mathcal{H}$, we slightly generalize their result as follows:

**Theorem 1.** Let $a$ and $b$ be self-adjoint operators on $\mathcal{H}$. Then the following are equivalent.

1. $ab = ba$.
2. $\exp(a + b) = \exp(a) \exp(b)$.
3. There exist a positive integer $n \geq 2$ and distinct non-zero real numbers $t_1, t_2, \ldots, t_{n-1}$ such that
   $$ (a + t_i b)^n = \sum_{k=0}^{n} \binom{n}{k} t_i^k a^{n-k} b^k $$
   for $i = 1, 2, \ldots, n - 1$.
4. There exist a positive integer $n \geq 2$ and distinct non-zero real numbers $t_1, t_2, \ldots, t_{n-1}$ such that
   $$ a^n - (t_i b)^n = (a - t_i b) \sum_{k=0}^{n-1} a^{n-k-1} (t_i b)^k $$
   for $i = 1, 2, \ldots, n - 1$.

DePrima and Richard([2]), and Uchiyama([11],[12]) independently prove that, for any positive operators $a$ and $b$, the following conditions are equivalent:

1. $ab = ba$.
2. $ab^n + b^n a$ is positive for all $n \in \mathbb{N}$.

We give a little weakened condition for two operators commuting.