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ABSTRACT. For two bounded positive linear operators a, b on a Hilbert space, we give conditions which imply the commutativity of a, b. Some of them are related to well-known formulas for indefinite elements, e.g., $(a+b)^n = \sum_k \binom{n}{k} a^{n-k} b^k$ etc. and others are related to the property of operator monotone functions. We also give a condition which implies the commutativity of a C*-algebra.

1. INTRODUCTION

Ji and Tomiyama ([5]) give a characterization of commutativity of C*-algebra, where they also give a condition that two positive operators commute. For bounded linear operators on a Hilbert space $\mathcal{H}$, we slightly generalize their result as follows:

Theorem 1. *Let a and b be self-adjoint operators on $\mathcal{H}$. Then the following are equivalent.*

(1) $ab = ba$.
(2) $\exp(a+b) = \exp(a)\exp(b)$.
(3) *There exist a positive integer $n \geq 2$ and distinct non-zero real numbers $t_1, t_2, \ldots, t_{n-1}$ such that*

$$(a + t_i b)^n = \sum_{k=0}^{n} \binom{n}{k} t_i^k a^{n-k} b^k$$

for $i = 1, 2, \ldots, n-1$.
(4) *There exist a positive integer $n \geq 2$ and distinct non-zero real numbers $t_1, t_2, \ldots, t_{n-1}$ such that*

$$a^n - (t_i b)^n = (a - t_i b) \sum_{k=0}^{n-1} a^{n-k-1} (t_i b)^k$$

for $i = 1, 2, \ldots, n-1$.

DePrima and Richard([2]), and Uchiyama([11],[12]) independently prove that, for any positive operators a and b, the following conditions are equivalent:

(1) $ab = ba$.
(2) $ab^n + b^n a$ is positive for all $n \in \mathbb{N}$.

We give a little weakened condition for two operators commuting.