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# CONVERGENCE RATE TO THE NONLINEAR WAVES FOR VISCOUS CONSERVATION LAWS ON THE HALF LINE*

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**Abstract.** We study the convergence rate of solutions to the initial-boundary value problem for scalar viscous conservation laws on the half line. Especially, we deal with the case where the Riemann problem for the corresponding hyperbolic equation admits transonic rarefaction waves. In this case, it is known that the solution tends toward a linear superposition of the stationary solution and the rarefaction wave. We show that the convergence rate is $(1+t)^{-\frac{1}{2}(1-\frac{1}{p})}\log^2(2+t)$ in $L^p$ norm $(1 \le p < \infty)$ and $(1+t)^{-\frac{1}{2}+\epsilon}$ in $L^\infty$ norm if the initial perturbation from the corresponding superposition is located in $H^1 \cap L^1$. The proof is given by a combination of the weighted $L^p$ energy method and the $L^1$ estimate.

**Key words.** Viscous conservation laws, convergence rate, weighted energy method, nonlinear waves, half space.

**AMS subject classifications.** 35L65, 35F30

**1. Introduction.** We consider the following scalar viscous conservation laws on the half line:

$$\begin{cases} u_t + f(u)_x = u_{xx}, & x > 0,\ t > 0, \\ u(0,t) = u_-, & t > 0, \\ \lim\limits_{x\to\infty} u(x,t) = u_+, & t > 0, \\ u(x,0) = u_0(x), & x > 0, \end{cases} \tag{1.1}$$

where the flux $f = f(u)$ is a given smooth function of $u$ satisfying $f(0) = f'(0) = 0$ and $u_\pm$ are given constants. In this problem, we assume that the initial function $u_0(x)$ satisfies $u_0(0) = u_-$ and $\lim_{x\to\infty} u_0(x) = u_+$ as the compatibility conditions. Throughout this paper, we impose the following condition on the flux $f(u)$: Either

$$f''(u) > 0 \quad \text{for} \quad u \in \mathbb{R}, \tag{1.2}$$

or

$$f''(0) > 0, \qquad f(u) > 0 \quad \text{for} \quad u \in [u_-, 0). \tag{1.3}$$

The main purpose of the present paper is to obtain the convergence rate under the boundary condition $u_- < 0 < u_+$ and the flux condition (1.2) or (1.3).

It is known that the asymptotic behavior of solutions to (1.1) is closely related to the solution of the Riemann problem for the corresponding hyperbolic equation (c.f. [5], [6]):

$$\begin{cases} u_t + f(u)_x = 0, \quad x \in \mathbb{R},\ t > -1, \\ u(x,-1) = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0. \end{cases} \end{cases} \tag{1.4}$$

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