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NONEXISTENCE OF POSITIVE FINITE MORSE INDEX SOLUTIONS TO AN ELLIPTIC PROBLEM WITH SINGULAR NONLINEARITY*

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Abstract. We consider the following elliptic problem

$$\Delta u = |y|^\alpha u^\tau \text{ in } \mathbb{R}^{\mathbb{N}},$$

where $N \geq 2, \alpha \geq 0$ and $\tau < 0$. We prove that if $2 \leq N < 2 + \frac{2(2+\alpha)}{\tau-1}(\tau - \sqrt{\tau(\tau-1)})$, there is no positive smooth solution with finite Morse index.

Key words. Positive solution, Finite Morse index, Singular nonlinearity.

AMS subject classifications. 35B45, 35J25

1. Introduction. We consider the positive smooth solutions of the following equation:

$$\Delta u = |y|^\alpha u^\tau \quad \text{in } \mathbb{R}^{\mathbb{N}}, \text{ with } N \geq 2, \alpha \geq 0, \tau < 0. \tag{1.1}$$

The motivation to study equation (1.1) comes the fact that this equation appears in many branches of applied science, such as in mechanics and in physics. In particular, it can be the steady states of thin films. In the literatures, the following equations

$$u_t = -\nabla \cdot (f(u)\nabla\Delta u) - \nabla \cdot (g(u)\nabla u) \tag{1.2}$$

have been used to model the dynamics of thin films of viscous fluids, where $u(x,t)$ is the height of the air (liquid) interface. The zero set $\sum_u = \{u = 0\}$ is the liquid (solid) interface and is sometimes called set of ruptures, which plays a very important role in the study of thin films. The coefficient $f(u)$ reflects surface tension effects, a typical choice is $f(u) = u^3$. And the coefficient $g(u)$ of the second-order term reflects additional forces such as gravity $g(u) = u^3$, and Van der Waals interactions $g(u) = u^m, m < 0$. For more information on the background of the equation (1.2), we refer to [2, 3, 19, 20] and the references therein. If we choose $f(u) = u^l, g(u) = u^m$, where $l, m \in \mathbb{R}$, and we consider the steady state of (1.2), we see that if u satisfies the following equation

$$u^l \nabla\Delta u + u^m \nabla u = \mathcal{C},$$

then u is a steady state of (1.2), where $\mathcal{C} = (C_1, C_2, ..., C_n)$ is some constant vector. By assuming $\mathcal{C} = 0$ (which prevents linear terms on x), we obtain

$$\Delta u + \frac{u^k}{k} - C = 0 \tag{1.3}$$

with $k = m - l + 1$ and C is some constant. Here we have assumed that $k \neq 0$, otherwise we replaced $\frac{u^k}{k}$ by $\ln u$. If we choose $f(u) = u^3, g(u) = -u^m$ with $m < 1$,

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