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REGULARITY OF THE EXTREMAL SOLUTION IN A MEMS MODEL WITH ADVECTION*

CRAIG COWAN† AND NASSIF GHOUSSOUB‡

Abstract. We consider the regularity of the extremal solution of the nonlinear eigenvalue problem

$$(S)_\lambda \qquad \begin{cases} -\Delta u + c(x)\cdot\nabla u &= \frac{\lambda}{(1-u)^2} \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ and $c(x)$ is a smooth bounded vector field on $\bar{\Omega}$. We show that, just like in the advection-free model ($c \equiv 0$), all semi-stable solutions are smooth if (and only if) the dimension $N \leq 7$. The novelty here comes from the lack of a suitable variational characterization for the semi-stability assumption. We overcome this difficulty by using a general version of Hardy's inequality. The same method applies for the Gelfand problem (i.e., exponential nonlinearity).

Key words. Extremal solution, regularity.

AMS subject classifications. 35J60, 35B32, 35D10, 35J20

1. Introduction.

The following equation has often been used to model a simple *Micro-Electro-Mechanical System* (MEMS) device:

$$(P)_\lambda \qquad \begin{cases} -\Delta u &= \frac{\lambda}{(1-u)^2} \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $\lambda > 0$ is proportional to the applied voltage and $0 < u(x) < 1$ denotes the deflection of the membrane. This model has been extensively studied, see [9], [10] in regards to the model and [6], [5], [7] for mathematical aspects of $(P)_\lambda$. It is well known (see above references) that there exists some positive finite critical parameter λ^* such that for all $0 < \lambda < \lambda^*$, the equation $(P)_\lambda$ has a smooth minimal stable (see below) solution u_λ, while for $\lambda > \lambda^*$ there are no weak solutions of $(P)_\lambda$ (see [6] for a precise definition of weak solution). Standard elliptic regularity theory yields that a solution u of $(P)_\lambda$ is smooth if and only if $\sup_\Omega u < 1$. One can also show that $\lambda \mapsto u_\lambda(x)$ is increasing and hence one can define the extremal solution

$$u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x),$$

which can be shown to be a weak solution of $(P)_{\lambda^*}$.

Recall that a smooth solution u of $(P)_\lambda$ is said to be *minimal* if any other solution v of $(P)_\lambda$ satisfies $u \leq v$ a.e. in Ω. Such solutions are then *semi-stable* meaning that the principal eigenvalue of the linearized operator

$$L_{u,\lambda} := -\Delta - \frac{2\lambda}{(1-u)^3}$$

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†Department of Mathematics, University of British Columbia, Vancouver, B.C. Canada V6T 1Z2 (ctcowan@math.ubc.ca). This work is part of the author's PhD dissertation in preparation under the supervision of N. Ghoussoub.

‡Department of Mathematics, University of British Columbia, Vancouver, B.C. Canada V6T 1Z2 (nassif@math.ubc.ca). Research partially supported by the Natural Science and Engineering Research Council of Canada.