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FURTHER REMARKS ON ASYMPTOTIC BEHAVIOR OF THE NUMERICAL SOLUTIONS OF PARABOLIC BLOW-UP PROBLEMS*

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Abstract. We consider a semilinear parabolic equation $u_t = u_{xx} + f(u)$ $(0 < x < 1,\ 0 < t)$, and a finite difference approximation for it. We discuss the way how the asymptotic profile of the blow-up solution is reproduced by the numerical solution. We will also determine qualitatively the influence of the definition of time mesh on the blow-up set of the numerical solution. Moreover, we show that explicit and implicit schemes may claim different blow-up sets.

Key words. Finite difference method, nonlinear parabolic equation, blow-up.

AMS subject classifications. 65M06

1. Introduction.

We consider the following semilinear parabolic equation

$$u_t(t,x) = u_{xx}(t,x) + f(u(t,x)) \qquad (0 < x < 1,\ 0 < t), \tag{1}$$

where the subscripts denote differentiation, with the initial and boundary conditions:

$$u(0,x) = u_0(x) \quad (0 < x < 1), \qquad u(t,0) = u(t,1) = 0 \quad (0 < t). \tag{2}$$

It is known (see [8, 10, 11]) that a solution with a large initial data blows up in finite time under certain growth assumptions on f as $u \to \infty$. Researches on parabolic blow-up problems like the present one have made considerable progress, and detailed knowledge on asymptotic profiles near the blow-up time, blow-up rate, complete and/or incomplete blow-up etc. have been established. See, for instance, [8], [9] or [15].

Compared with the theoretical study, numerical analysis of the blow-up problem does not seem to be explored enough. Our purpose in the present paper is to provide some mathematical analysis on finite difference approximations. Let us briefly recall some investigations of the past. Nakagawa[13] considered an explicit finite difference scheme with uniform spatial grids and adaptive step sizes in time. He showed that his numerical solutions converge to the solution up to the blow-up time. This implies not only that the numerical solutions converge in the time interval where the solution is smooth but also that the numerical blow-up time converges to the real blow-up time. If we recall that the convergence, in usual numerical analysis, is proved under some smoothness assumptions, this result may be remarkable, since regularity of the solution is lost at the blow-up time. His result was later improved substantially by [1, 2, 3]. Later, Chen[5] considered a similar problem and showed that the "blow-up set" of the numerical solution can be different from the blow-up set of the solutions of PDE. The problems which were left unanswered in these papers were dealt with in [6, 7], and many of them were solved. However, some questions were left in [7] as open questions, and we would like to shed light on these questions in the present paper.

The rest of the paper is organized as follows. In section 2, known results are recalled and problems to be addressed in the present paper will be explained. In

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