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ON BELLMAN'S EQUATIONS WITH VMO COEFFICIENTS*

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Abstract. We present a result about solvability in W_p^2, $p > d$, in the whole space $\mathbb{R}^d$ of Bellman's equations with VMO "coefficients". Parabolic equations are touched upon as well.

Key words. Vanishing mean oscillation, fully nonlinear equations, Bellman's equations.

AMS subject classifications. 35J60

1. Main result. Let $\mathbb{R}^d$ be the Euclidean space of points $x = (x^1, ..., x^d)$, $x^i \in \mathbb{R} = (-\infty, \infty)$. Fix a $\delta \in (0,1)$ and denote by $\mathcal{S}_\delta$ the set of symmetric $d \times d$-matrices $a = (a^{ij})$ satisfying

$$\delta|\xi|^2 \le a^{ij}\xi^i\xi^j \le \delta^{-1}|\xi|^2, \quad \forall \xi \in \mathbb{R}^d.$$

Let Ω be a separable metric space and assume that for any $\omega \in \Omega$ and $x \in \mathbb{R}^d$ we are given $a(\omega, x) \in \mathcal{S}_\delta$, $b(\omega, x) \in \mathbb{R}^d$, and $c(\omega, x), f(\omega, x) \in \mathbb{R}$. We assume that these functions are measurable in x for each ω, continuous in ω for each x, and

$$|b(\omega, x)| + c(\omega, x) \le K, \quad c(\omega, x) \ge 0, \quad \forall \omega, x,$$

$$\bar{f}(x) := \sup_{\omega \in \Omega} |f(\omega, x)| < \infty \quad \forall x,$$

where K is a fixed constant. Observe that, owing to the continuity of f in ω and separability of Ω, the function $\bar{f}$ is measurable. For $r > 0$ and $x \in \mathbb{R}^d$ set

$$B_r(x) = \{y \in \mathbb{R}^d : |x - y| < r\}, \quad B_r = B_r(0).$$

For a measurable set $\Gamma \subset \mathbb{R}^d$ by $|\Gamma|$ we denote its volume. In Section 4 we will use the same notation for measurable $\Gamma \subset \mathbb{R}^{d+1}$. Introduce,

$$(u)_\Gamma = \fint_\Gamma u(x)\,dx = \frac{1}{|\Gamma|}\int_\Gamma u(x)\,dx.$$

In particular,

$$(a)_{B_r(x)}(\omega) = \fint_{B_r(x)} a(\omega, y)\,dy$$

In the following assumption there is a parameter $\theta \in (0,1]$, whose value will be specified later.

ASSUMPTION 1.1. There exists an $R_0 \in (0, \infty)$ such that for any $r \in (0, R_0]$ and $x \in \mathbb{R}^d$ we have

$$\fint_{B_r(x)} \sup_{\omega \in \Omega} |a(\omega, y) - (a)_{B_r(x)}(\omega)|\,dy \le \theta. \tag{1.1}$$

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