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SPECTRAL ANALYSIS OF BIRTH–AND–DEATH PROCESSES WITH AND WITHOUT KILLING VIA PERTURBATION*

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Abstract. A population process with constant birth rates $\lambda_n \equiv \lambda$ and death rates $\mu_n \equiv \mu$ is supplied with strong linear killing rates $\gamma_n = \gamma n$ for $n \in \mathbb{N}_0$. The process is analyzed in view of its spectral representation: The underlying orthogonal polynomials are seen to be Lommel polynomials $R_{n,\nu}(x)$, considered as functions in their parameter ν. Regarding the Stieltjes transform of their orthogonality measure, we recognize that it is purely discrete and given by the zeros of a Bessel function $J_{\nu-1}(x)$ in its order. Qualitative and quantitative results for the zeros are worked out via methods from the theory of Bessel functions and from regular perturbation theory. The same ideas are applied successfully to study the spectrum for associated Meixner polynomials, corresponding to linear birth and death rates.

Key words. Birth–and–death processes with killing, spectral representation, Stieltjes transform, Lommel polynomials, Bessel functions, regular perturbation theory, associated Meixner polynomials.

AMS subject classifications. Primary 33C45, 60J80; Secondary 33C10, 47A55

1. Introduction: New killing structures.

The study of **birth–and–death processes with killing** was initiated by S. Karlin and S. Tavaré in [8]. A general treatment of such structures from a spectral point-of-view has recently been worked out by E. van Doorn and A. Zeifman in [16]: Given birth rates λ_n, death rates μ_n and killing rates γ_n at any state $n \in \mathbb{N}_0$, the transition probabilities of the process $\mathcal{X}$ after time $t \geq 0$ may be expressed as

$$\mathbb{P}\big(\mathcal{X}(t+t_0) = m \,\big|\, \mathcal{X}(t_0) = n\big) \equiv \mathbb{P}_{nm}(t) = \frac{1}{\pi_n}\int_0^\infty e^{-xt} F_n(x)F_m(x)\, \mathrm{d}\pi(x) \qquad \forall\, n,m \in \mathbb{N}_0. \tag{1}$$

Here $\{F_n \mid n \in \mathbb{N}_0\}$ is a set of **orthogonal polynomials (OP)** with a corresponding probability measure π supported on $[0,\infty)$. They satisfy the recurrence relation

$$\mu_{n+1}F_{n+1}(x) + \lambda_{n-1}F_{n-1}(x) = (\lambda_n + \mu_n + \gamma_n - x)F_n(x) \quad \forall\, n \in \mathbb{N}_0, \tag{2}$$

initialized by $F_0 = 1$ and $F_{-1} = 0$, while the coefficients π_n are explicitly given as $\pi_n = \prod_{k=1}^n \frac{\lambda_{k-1}}{\mu_k}$ for $n \in \mathbb{N}_0$. Despite this somewhat easy access, up to now there have only been two (nontrivial) processes with killing whose spectral representation could be obtained explicitly:

1. The **linear** process with killing, having rates $\lambda_n = \lambda(n+b)$, $\mu_n = \mu n$ and $\gamma_n = \gamma n$ for $n \in \mathbb{N}_0$, was solved in [8]; but its solution leads back to the well-known Meixner polynomials.
2. The special **quadratic** process with rate structure $\lambda_n = 2(n+1)(2n+1)$, $\mu_n = 2n(2n+1)$ and $\gamma_n = 2a(2n+1)^2$ was considered in [5]; it leads to new OP of an elliptic-like character.

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