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# ELLIPTIC AND PARABOLIC EQUATIONS WITH MEASURABLE COEFFICIENTS IN $L_P$-SPACES WITH MIXED NORMS*

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**Abstract.** The unique solvability results for second order parabolic and elliptic equations in Sobolev spaces with mixed norms are presented. The second order coefficients are measurable in one spatial variable and VMO (vanishing mean oscillation) in the other spatial variables. In the parabolic case, the coefficients (except $a^{11}$) are further allowed to be only measurable in time. We first prove the solvability results for equations in the whole Euclidean space. Then, using these results as well as some extension techniques, we prove the solvability results for equations on a half space without any boundary estimates. The mixed norms we present here are more general than the usual mixed norm $L_q^t L_p^x$.

**Key words.** Second order elliptic and parabolic equations, vanishing mean oscillation, Sobolev spaces with mixed norms.

**AMS subject classifications.** 35J15, 35K10, 35R05, 35A05

**1. Introduction.** In this paper we study elliptic and parabolic equations of non-divergence type in Sobolev spaces with mixed norms. The differential equations we consider are (elliptic and parabolic, respectively)

$$a^{ij}(x)u_{x^i x^j} + b^i(x)u_{x^i} + c(x)u = f, \tag{1}$$

$$u_t + a^{ij}(t,x)u_{x^i x^j} + b^i(t,x)u_{x^i} + c(t,x)u = f. \tag{2}$$

The equations are assumed to be uniformly non-degenerate with bounded coefficients. The regularity assumptions on the coefficients $a^{ij}$ (no regularity assumptions are needed for coefficients $b^i$ and $c$) are, roughly speaking, that they are merely measurable (i.e., no regularity assumptions) in one spatial direction, and belong to the space of VMO (mean vanishing oscillation) as functions of the other variables. In the parabolic case, they (except $a^{11}$) are measurable in two variables including the time variable and in the space of VMO as functions of the remaining variables. The coefficient $a^{11}$ is measurable in one spatial variable and VMO in the other variables.

There has been, in fact, considerable study of *parabolic* equations in mixed norm spaces in the literature (see, e.g., [4, 18, 11, 10, 12, 21, 3, 6, 20, 14] and references therein). The usual mixed norms are of the form $L_q((0,T), L_p)$, that is, $q$ summability in the time variable and $p$ summability in the spatial variables. For example, in [12, 21, 20, 14] parabolic equations as in (2) (quasi-linear equations in [20]) are investigated in Sobolev spaces with the mixed norm $L_q((0,T), L_p(\Omega))$, where $\Omega \subseteq \mathbb{R}^d$. In [3] one sees parabolic systems in $L_q((0,T), X)$, where $X$ is an $L_p$ space with a Muckenhoupt weight. In this paper, however, by mixed norms we mean, not the usual mixed norms, but more general ones defined as follows.

First we explain some notations. By $\mathbb{R}^d$ we mean a $d$-dimensional Euclidean space and a point in $\mathbb{R}^d$ is denoted by $x$. In the parabolic case we consider $\mathbb{R}^{d+1}$, where a point in $\mathbb{R}^{d+1}$ is denoted by $(t,x)$. Throughout the paper, we fix two nonnegative

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