METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 15, No. 4, pp. 427–436, December 2008

002

# GENERALIZING VARIATIONAL THEORY TO INCLUDE THE INDEFINITE INTEGRAL, HIGHER DERIVATIVES, AND A VARIETY OF MEANS AS COST VARIABLES*

JOHN GREGORY†

**Abstract.** In this paper we generalize the Calculus of Variations setting from the arguments $\big(t, x(t), x'(t)\big)$ to the arguments $\big(t, x(t), x'(t), \hat{x}(t)\big)$, where $\hat{x}(t)$ is any indefinite integral of $x(t)$ or its extension. Related results are to various types of "averaging" variables and to higher derivatives.

Of interest is that this process extends the basic necessary condition, the Euler equation(s), from a second order ordinary differential equation (ODE) to a second order differential equation involving indefinite integrals. These ideas can be extended to optimal control and other constraint optimization problems whose trajectory includes $\hat{x}$ and/or averaging operators.

Of special interest is that these results can be obtained either by using a new extension of classical arguments or by a new use of the author's constraint optimization theory. Finally, these multi-integral problems can now be solved by efficient numerical methods, previously developed by the author, with a global a priori error estimate of $O(h^2)$.

**Key words.** Generalized variational theory, new dependent variables, double indefinite integrals.

**AMS subject classifications.** 49J15, 49K15, 34K35, 65K10

## 1. Introduction.

The main purpose of this paper is to extend the ideas and methods of the calculus of variations/optimal control theory to problems which include the antiderivative and related dependent variables. Our basic problem is given as

$$\begin{aligned} &\min \int_a^b f\big(t, x(t), x'(t), \hat{x}(t)\big)\, dt \\ &\text{s.t. } x(a) = x_a \text{ and } x(b) = x_b \text{ where} \\ &\hat{x}(t) = \int_a^t x(t)\, dt. \end{aligned} \tag{1.1}$$

For a multitude of reasons we also assume that $f_{x'x'} > 0$ (see, for example, [1], [5] or [8]).

We assume this problem has a unique solution. "Smoothness" conditions will be obvious and assumed as needed. The initial conditions $x(a)$ and $x(b)$ can be replaced by the usual transversality conditions if these values are unspecified.

We note that our solution for (1.1) gives a more general theory for the usual classical calculus of variations/optimal control theory. In particular, the major necessary condition, the Euler equation, is no longer an ordinary second order differential equation but, now, an equation with indefinite integrals. This often implies the equation is essentially a third or fourth order ODE.

In Sections 2 and 3 we derive our key results using two distinct methods. In part this is due to the fact that our methods are themselves new and these results are so new that we wish to double check them. An example is given at the end of Section 2 which is then revisited after Section 3.

In Section 4 we extend these ideas to optimal control theory where both the cost functional and the trajectory equation include the new variable, $\hat{x}$. We will show that the Pontryagin Maximal Principle is a special case of our results.

*Received May 31, 2007; accepted for publication October 14, 2008.
†Department of Mathematics, Mailcode 4408, Southern Illinois University Carbondale, 1245 Lincoln Drive, Carbondale IL 62901, USA (jgregory@math.siu.edu).