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A NEW REGULARITY CRITERION FOR THE NAVIER-STOKES EQUATIONS IN TERMS OF THE GRADIENT OF ONE VELOCITY COMPONENT *

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Abstract. In this paper we consider the regularity criteria for the weak solutions to the Navier-Stokes equations in $\mathbb{R}^3$. It is proved that if the gradient of any one component of the velocity field belongs to $L^{\alpha,\gamma}$ with $2/\alpha + 3/\gamma = 3/2$, $3 \leq \gamma < \infty$, then the weak solution actually is strong.

1. Introduction. We consider the following Cauchy problem for the incompressible Navier-Stokes equations in $\mathbb{R}^3 \times (0, T)$

$$\begin{cases} \dfrac{\partial u}{\partial t} + u \cdot \nabla u + \nabla p = \Delta u, \\ \text{div} u = 0, \\ u(x, 0) = u_0(x), \end{cases} \tag{1}$$

where $u = (u_1(x,t), u_2(x,t), u_3(x,t))$ is the velocity field, $p(x,t)$ is a scalar pressure, and $u_0(x)$ with $\text{div} u_0 = 0$ in the sense of distribution is the initial velocity field.

The study of the incompressible Navier-Stokes equations in three space dimensions has a long history. In the pioneering work [10] and [7], Leray and Hopf proved the existence of its weak solutions $u(x,t) \in L^\infty(0,T; L^2(\mathbb{R}^3)) \cap L^2(0,T; H^1(\mathbb{R}^3))$ for given $u_0(x) \in L^2(\mathbb{R}^3)$. But the uniqueness and regularity of the Leray-Hopf weak solutions are still big open problems. In [12], Scheffer began to study the partial regularity theory of the Navier-Stokes equations. Deeper results were obtained by Caffarelli, Kohn and Nirenberg in [2]. Further result can be found in [17] and references there in.

On the other hand, the regularity of a given weak solution u can be shown under additional conditions. In 1962, Serrin [13] proved that if u is a Leray-Hopf weak solution belonging to $L^{\alpha,\gamma} \equiv L^\alpha(0,T; L^\gamma(\mathbb{R}^3))$ with $2/\alpha + 3/\gamma \leq 1, 2 < \alpha < \infty, 3 < \gamma < \infty$, then the solution $u(x,t) \in C^\infty(\mathbb{R}^3 \times (0,T))$. From then on, there are many criterion results added on u. In [18] and [5], von Wahl and Giga showed that if u is a weak solution in $C([0,T); L^3(\mathbb{R}^3))$, then $u(x,t) \in C^\infty(\mathbb{R}^3 \times (0,T))$; Struwe [16] proved the same regularity of u in $L^\infty(0,T; L^3(\mathbb{R}^3)$ provided $\sup_{0<t<T} \|u(x,t)\|_{L^3}$ is sufficiently small and Kozono and Sohr [8] obtained the regularity for the weak solution $u(x,t) \in C^\infty(\mathbb{R}^3 \times (0,T))$ provided $u(x,t)$ is left continuous with respect to L^3-norm for every $t \in (0,T)$. Recently Kozono and Taniuchi [9] showed that if a Leray-Hopf weak solution $u(x,t) \in L^2(0,T; BMO)$, then $u(x,t)$is actually a strong solution of (1) on $(0,T)$. $L^{\alpha,\gamma}$ is defined by

$$\|u\|_{L^{\alpha,\gamma}} = \begin{cases} \left(\displaystyle\int_0^t \|u(.,\tau)\|_{L^\gamma}^\alpha d\tau\right)^{1/\alpha} & \text{if } 1 \leq \alpha < \infty \\ \operatorname{ess}\sup\limits_{0<\tau<t} \|u(.,\tau)\|_{L^\gamma} & \text{if } \alpha = \infty \end{cases}$$

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