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ON SOLUTIONS WITH POINT RUPTURES FOR A SEMILINEAR ELLIPTIC PROBLEM WITH SINGULARITY*

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Abstract. We consider the following semilinear elliptic equation with singular nonlinearity:

$$\Delta u - \frac{\lambda}{u^\nu} = 0 \text{ in } B,\ u = \psi \text{ on } \partial B$$

where $\lambda > 0, \nu > 0$ and $\psi \in C^{2,\alpha}(\partial B)$ and B is the unit ball in $\mathbb{R}^N$. Under various conditions on λ, ν and ψ, we construct solutions with one isolated zero in B.

Key words. Point ruptures, singularity.

AMS subject classifications. Primary 35B35, 35B40; Secondary 35J60

1. Introduction. Let B be the unit ball of $\mathbb{R}^N$ ($N \geq 2$). The main purpose of this paper is to construct nonnegative solutions with one isolated zero point of the semilinear elliptic Dirichlet problem

$$\Delta u - \lambda u^{-\nu} = 0 \quad \text{in } B, \quad u = \psi \ \text{ on } \partial B, \tag{1.1}$$

where $\lambda,\ \nu > 0$, $\psi \in C^{2,\alpha}(\partial B)$ with $\psi(\theta) > 0$ for $\theta \in S^{N-1} = \partial B$.

Problem (1.1) appears in several applications in mechanics and physics, and in particular can be used to model the electrostatic Micro-Electromechanic System (MEMS) devices. See [FMP], [GG1], [GG2], [GG3], [GPW] and the references therein. In particular, in [GG1], [GG2] and [GG3], Ghoussoub and Guo have given a thorough study on the following problem

$$\begin{cases} u_t = \Delta u - \frac{\lambda f(x)}{u^2}, & x \in \Omega, t > 0, \\ u(x,0) = 1 \text{ for } \ x \in \Omega,\ u(x,t) = 1 \text{ for } x \in \partial\Omega \end{cases} \tag{1.2}$$

where $\lambda > 0, f(x)$ is a positive function and Ω is a bounded smooth domain in $\mathbb{R}^N$. (1.1) is just the steady state of (1.2) with $f(x) \equiv 1$ and $\nu = 2$. The set $\{x|u(x) = 0\}$ is called **touch town** set and plays an important role in MEMS.

Problem (1.1) can also be considered as steady state problem of thin films problems. Equations of the type

$$u_t = -\nabla \cdot (f(u)\nabla\Delta u) - \nabla \cdot (g(u)\nabla u) \tag{1.3}$$

have been used to model the dynamics of thin films of viscous fluids, where $z = u(x,t)$ is the height of the air/liquid interface. The zero set $\mathbf{\Sigma}_u = \{u = 0\}$ is the liquid/solid interface and is sometimes called set of **ruptures**. Ruptures play a very important role in the study of thin films. The coefficient $f(u)$ reflects surface tension effects- a typical choice is $f(u) = u^3$. The coefficient of the second-order term can reflect additional forces such as gravity $g(u) = u^3$, van der Waals interactions $g(u) = u^m, m < 0$. For more background on thin films, we refer to [BBD, BP1, BP2, LP1, LP2, LP3, WB,

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