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RESULTS ON POSITIVE SOLUTIONS OF ELLIPTIC EQUATIONS WITH A CRITICAL HARDY-SOBOLEV OPERATOR*

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Dedicated to Neil Trudinger with respect and friendship on the occasion of his sixty fifth birthday

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1. Introduction.

Let $N \geq 3, 1 \leq k < N$ and $\mathbb{R}^N = \mathbb{R}^k \times \mathbb{R}^{N-k}$. Write $x = (y, z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$. We are concerned with classifying non-negative solutions of

$$-\Delta u = \frac{u^{2^*(t)-1}(x)}{|y|^t}, \qquad x \in \mathbb{R}^N, \tag{1.1}$$

where $0 < t < \min\{2, k\}$, $2^*(t) = \frac{2(N-t)}{N-2}$.

We will use $D^{1,p}(\mathbb{R}^N)$, $1 \leq p < N$, to denote the completion of $C_c^\infty(\mathbb{R}^N)$, the set of C^∞ functions with compact support in $\mathbb{R}^N$, under the norm $\|u\|_{D^{1,p}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\nabla u|^p\right)^{\frac{1}{p}}$. By the Gagliado-Nirenberg-Sobolev inequality,

$$\|u\|_{L^{\frac{pN}{N-p}}(\mathbb{R}^N)} \leq C(N,p)\|u\|_{D^{1,p}(\mathbb{R}^N)}.$$

Thus we use $D^{1,2}_{loc}(\mathbb{R}^N)$ to denote those functions u which satisfy, on all compact subsets K of $\mathbb{R}^N$, $u \in L^{\frac{2N}{N-2}}(K)$ and $\nabla u \in L^2(K)$. It is the same as $H^1_{loc}(\mathbb{R}^N)$, another standard notation which denotes the set of functions u satisfying $u, \nabla u \in L^2(K)$ for all compact subsets K of $\mathbb{R}^N$.

A $D^{1,2}_{loc}(\mathbb{R}^N)$ solution of (1.1) is in L^∞_{loc}. This can be proved by arguments similar to those used by Trudinger in [19] in proving the L^∞ regularity of H^1 solutions to the Yamabe equation, see [8] and [16] where Hölder regularity of solutions of (1.1) were also studied. Clearly a positive solution u of (1.1) is C^∞ in $\{(y,z) \mid y \neq 0\}$. See [1], [4], [5], [7], [8], [10], [11], and the references therein for related studies.

Since u is superharmonic, non-negative and nonzero, it follows from the maximum principle (see, e.g. [13]) that

$$\inf_{|x|\leq 2} u(x) > 0, \qquad \inf_{|x|\geq 1} (|x|^{N-2}u(x)) > 0. \tag{1.2}$$

Our first result is

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