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MONOTONE MAPS OF $\mathbb{R}^n$ ARE QUASICONFORMAL*

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For Neil Trudinger

Abstract. We give a new and completely elementary proof showing that a δ–monotone mapping of $\mathbb{R}^n$, $n \geq 2$ is K–quasiconformal with linear distortion

$$K \leq \frac{1+\sqrt{1-\delta^2}}{1-\sqrt{1-\delta^2}}$$

This sharpens a result due to L. Kovalev.

Key words. Monotone mapping, quasiconformal.

AMS subject classifications. 30C60

1. Introduction. In [?] L.V. Kovalev proved the interesting fact that a δ–monotone mapping of $\mathbb{R}^n$ is K–quasiconformal for some distortion constant K depending only on δ. Here we give a new poof of this result using methods which are rather more elementary than those employed in [?], going through a compactness argument which is more or less standard in the theory of quasiconformal mappings. We are also able to give the precise estimates relating the monotonicity constant δ and the distortion constant K (these precise estimates were already given in two dimensions in our earlier work [?].) We remark that the proof given here works without modification for monotone mappings of Hilbert spaces.

Let us recall the relevant definitions. A function $h : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is called *δ–monotone*, $0 < \delta \leq 1$ if for every $z, w \in \Omega$

$$\langle h(z) - h(w), z - w\rangle \geq \delta |h(z) - h(w)||z - w| \tag{1}$$

There is no supposition of continuity here. It is obvious from the definition at (1) that the family of δ–monotone maps is invariant under rescaling and translation. Of course $\langle h(z) - h(w), z - w\rangle = |h(z) - h(w)||z - w| \cos(\alpha)$ where α is the angle between these vectors. Thus δ–monotone maps are prevented from rotating the vector formed from a pair of points more than an angle $|\arccos(\delta)| < \pi/2$. Monotone mappings have found wide application in partial differential equations for decades, particularly those second order PDEs of divergence type, because of the well known Minty-Browder theory [?, ?]. Roughly the monotonicity condition is used to bound a nonlinear operator away from a curl. See the monograph [?] for some of this theory and connections with quasiconformal mappings and second order nonlinear divergence equations in the plane. This brings us to our next definition. An orientation preserving injection

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