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CONSTRUCTION OF THE PARALLEL TRANSPORT IN THE WASSERSTEIN SPACE*

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Abstract. In this paper we study the problem of parallel transport in the Wasserstein spaces $\mathscr{P}_2(\mathbb{R}^d)$. We show that the parallel transport exists along a class of curves whose velocity field is sufficiently smooth, and that we call regular. Furthermore, we show that the class of regular curves is dense in the class of absolutely continuous curves and discuss the problem of parallel transport along geodesics. Most results are extracted from the PhD thesis [8].

Key words. Optimal transport, Wasserstein space, curvature tensor.

AMS subject classifications. 28A33, 35K55, 47J35

1. Introduction.

In the last few years, starting from the seminal papers [14, 4, 12, 9], the geometric and differential properties of the space $\mathscr{P}_2(\mathbb{R}^d)$ of probability measures in $\mathbb{R}^d$ with finite quadratic moments, endowed with the quadratic optimal transportation distance, have been deeply investigated. Motivations for this analysis come from PDE's, Functional Inequalities, Riemannian Geometry. We refer to [16] for a comprehensive presentation of this wide and continuously expanding research field.

A complete theory of the first-order differential properties of $\mathscr{P}_2(\mathbb{R}^d)$ has been estabilished in [1] (starting from the heuristics developed in [14]), without any extra regularity assumption, either on the measures involved, or on the velocity fields. These results lead to a complete theory of gradient flows in $\mathscr{P}_2(\mathbb{R}^d)$ which extends, as a matter of fact, also to the case when $\mathbb{R}^d$ is replaced by more general spaces (see for instance [3, 13, 15]), for instance an infinite-dimensional Hilbert space. We recall the basic facts of the first-order theory in Section 2.

On the other hand, much less is known on the second-order properties of $\mathscr{P}_2(\mathbb{R}^d)$: the only paper we are aware of is [10], where the parallel transport equation and the curvature tensor of $\mathscr{P}(M)$ are computed, mostly at a formal level, when M is a compact Riemannian manifold; in Section 7 we borrow some computations of the sectional curvature of $\mathscr{P}_2(\mathbb{R}^d)$ from [10].

In this paper, whose content is essentially extracted from Chapter 6 of [8], we focus on some analytic aspects: we introduce a class of curves μ_t in $\mathscr{P}_2(\mathbb{R}^d)$ along which the parallel transport of tangent vectors can be defined. In the case when $\mu_t = \rho_t \mathscr{L}^d$ ($\mathscr{L}^d$ being the Lebesgue measure), the PDE corresponding to the parallel transport of a gradient vector field $\nabla\varphi_t$ is, in accordance with [10],

$$\nabla \cdot \Big((\partial_t \nabla\varphi_t + \nabla^2\varphi_t \cdot v_t)\rho_t \Big) = 0. \tag{1.1}$$

Existence and uniqueness for this evolution problem can presumably studied by direct PDE methods, although difficulties obviously are due to the degeneracy of ρ_t, which results in a lack of uniform ellipticity. Moreover, additional difficulties appear if one is willing to consider unbounded densities ρ_t, and even (in the same spirit of the theory in [1]) measures μ_t that have a singular part with respect to $\mathscr{L}^d$. For these

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