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LOG IMPROVEMENT OF THE PRODI-SERRIN CRITERIA FOR NAVIER-STOKES EQUATIONS*

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Abstract. This article is devoted to a Log improvement of Prodi-Serrin criterion for global regularity to solutions to Navier-Stokes equations in dimension 3. It is shown that the global regularity holds under the condition that $|u|^5/(\log(1+|u|))$ is integrable in space time variables.

Key words. Navier-Stokes, regularity criterion, a priori estimates

AMS subject classifications. 35B65, 76D03, 76D05

1. Introduction. In this article, we consider the Navier-Stokes equation on $\mathbb{R}^3$, given by

$$\partial_t u - \triangle u + \operatorname{div}(u \otimes u) + \nabla p = 0, \tag{1}$$

$$\operatorname{div}(u) = 0, \tag{2}$$

where u is a vector-valued function representing the velocity of the fluid, and p is the pressure. Note that the pressure depends in a non local way on the velocity u. It can be seen as a Lagrange multiplier associated to the incompressible condition (2). The initial value problem of the above equation is endowed with the condition that $u(0,\cdot) = u_0 \in L^2(\mathbb{R}^3)$. Leray [15] and Hopf [10] had already established the existence of global weak solutions for the Navier-Stokes equation. In particular, Leray introduced a notion of weak solutions for the Navier-Stokes equation, and proved that, for every given initial datum $u_0 \in L^2(\mathbb{R}^3)$, there exists a global weak solution $u \in L^\infty(0,\infty; L^2(\mathbb{R}^3)) \cap L^2(0,\infty; \dot{H}^1(\mathbb{R}^3))$ verifying the Navier-Stokes equation in the sense of distribution. From that time on, much effort has been devoted to establish the global existence and uniqueness of smooth solutions to the Navier-Stokes equation. Different Criteria for regularity of the weak solutions have been proposed. The Prodi-Serrin conditions (see Serrin [20], Prodi [18], and [21]) states that any weak Leray-Hopf solution verifying $u \in L^p(0,\infty; L^q(\mathbb{R}^3))$ with $2/p + 3/q = 1$, $2 \le p < \infty$, is regular on $(0,\infty) \times \mathbb{R}^3$. Notice that if $p = q$, this corresponds to $u \in L^5((0,\infty) \times \mathbb{R}^3)$. The limit case of $L^\infty(0,\infty; L^3(\mathbb{R}^3))$ has been solved very recently by L. Escauriaza, G. Seregin, and V. Sverak (see [11]). Other criterions have been later introduced, dealing with some derivatives of the velocity. Beale Kato and Majda [1] showed the global regularity under the condition that the vorticity $\omega = \operatorname{curl} u$ lies in $L^\infty(0,\infty; L^1(\mathbb{R}^3))$ (see Kozono and Taniuchi for improvement of this result [13]). Beirão da Veiga show in [2] that the boundedness of ∇u in $L^p(0,\infty; L^q(\mathbb{R}^3))$ for $2/p + 3/q = 2$, $1 < p < \infty$ ensures the global regularity. In [7], Constantin and Fefferman gave a condition involving only the direction of the vorticity. Improvements of this results were obtained by Beirão da Veiga [3, 4, 5, 6], and Zhou [24, 25]. Let us also cite a condition involving the lower bound of the pressure introduced by Seregin and Sverak in [19], and conditions involving only one of the component of u (see Penel and Pokorny [17], He [9], and Zhou [23, 26]).

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