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EXISTENCE OF SOLUTIONS TO THE THREE DIMENSIONAL BAROTROPIC-VORTICITY EQUATION*

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Abstract. We prove existence of maximizers for a variational problem in $\mathbb{R}^3_+$. Solutions represent steady geophysical flows over a surface of variable height which is bounded from below.

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AMS subject classifications. 35J60, 76B03, 76B47, 76M30, 76U05

1. Introduction. In this paper we prove existence of maximizers for a variational problem which describes a geophysical flow over a surface of variable height, bounded from below, such as a seamount in the ocean or a mountain in the atmosphere. The basic equation governing such flows is the three dimensional barotropic vorticity equation given by

$$[\psi, \zeta] = 0,$$

where $[.,.]$ denotes the Jacobian and ψ represents the stream function, $-\zeta$ the potential vorticity given by

$$-\zeta = \Delta\psi + h,$$

where h is the height of the bottom surface.

In [6] and [9] similar problems have been considered in two dimensions. Here, the problem has been formulated in three dimensions which is more realistic. In addition, from a technical point of view, due to drastic differences between the fundamental solutions of $-\Delta$ in two and three dimensions the estimates in [6] and [9] are not applicable. In particular we single out the simple but crucial result stated in Lemma 6 in section 3.

To prove the existence we follow the method proposed by Benjamin [3]. To do this we begin by considering the variational problem over half spheres. In order to prove existence of maximizers in this situation we employ the technology extensively developed by Burton [4,5]. Then using a limiting argument we show that maximizers for *large* half spheres indeed are maximizers for the original problem; the radius of the critical half sphere turns out to be the radius of the smallest two dimensional disc containing the support of the height function h.

2. Definitions and notations. Henceforth we assume $p \in (3, \infty)$. The ball centered at $x \in \mathbb{R}^3$ with radius R is denoted $B_R(x)$; in particular when the center is the origin we write B_R. For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, we write $\bar{x} = (x_1, x_2, -x_3)$ and we define $\mathbb{R}^3_+ = \{x \in \mathbb{R}^3 : x_3 > 0\}$. For a measurable set $A \subseteq \mathbb{R}^3$, $\mid A \mid$ denotes the three dimensional Lebesgue measure of A. If A is measurable, then $x \in A$ is called a density point of A whenever $\mid B_\varepsilon(x) \bigcap A \mid > 0$, for all positive ε. The set of all density

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