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SEMICLASSICAL FORMS OF CLASS $s = 2$: THE SYMMETRIC CASE, WHEN $\Phi(0) = 0$*

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Abstract. A regular linear form u is said to be semiclassical, if there exist two polynomials Φ monic and Ψ, $\deg(\Psi) \geq 1$ such that $(\Phi u)' + \Psi u = 0$. Recently, all the symmetric semiclassical linear forms of class $s \leq 1$ are determined. In this paper, by considering the inverse problem of the product of a form by a polynomial in the square case, we carry out the complete description of the symmetric semiclassical linear forms of class $s = 2$, when $\Phi(0) = 0$ which generalize those of class $s = 1$. Essentially, three canonical cases appear. Some particular cases refer to well-known orthogonal sequences. Representations of these linear forms are given.

Key words. Orthogonal polynomials, symmetric forms, semiclassical forms, integral representations.

AMS subject classifications. 42C05, 33C45

Introduction. Semiclassical orthogonal polynomials were introduced in [23]. They are natural generalization of the classical polynomials. Maroni [19,21] has worked on the linear form of moments and has given a unified theory of this kind of polynomials. A semiclassical linear form u satisfies the distributional equation $(\Phi u)' + \psi u = 0$ where $\Phi(x)$ is a monic polynomial and $\Psi(x)$ is a polynomial with $\deg(\Psi) \geq 1$. In [2], the authors determine all the symmetric semiclassical linear forms of class $s = 1$. See also [3,4] for some special cases. It is natural to consider the problem of determining all the symmetric semiclassical linear forms of class $s = 2$. In this paper, we are interested in the case when $\Phi(0) = 0$. For this, we consider the inverse problem of the product of a linear form by a polynomial by studying the following problem: given a symmetric semiclassical linear form v, find the symmetric linear form u defined by $x^2 u = -\lambda v \Leftrightarrow u = -\lambda x^{-2} v + \delta_0$, $\lambda \in \mathbb{C}^*$ in a different way than [17,1]. This kind of problem is an interesting process to construct certain families of semiclassical polynomials as treated in many recent works ([1], [5], [16], [17], [22]). The first section is devoted to the preliminary results and notations used in the sequel. In the second section, We found a relation between the symmetric semiclassical linear forms of class $s = 2$ and those of class $s \leq 1$ (Theorem 2.3.). Using this relation, we give, in Section 2, all the linear forms which we look for. Three canonical cases for the polynomial Φ arise: $\Phi(x) = x^2$, $\Phi(x) = x^4$ and $\Phi(x) = x^2(x^2 - 1)$. Representations of the new linear forms are obtained. As it turned out, we obtained explicitly three nonsymmetric semiclassical linear forms of class $s = 1$.

1. Notations and preliminary results. Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its topological dual. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n\rangle, n \geq 0$, the moments of u. For any linear form u and any polynomial h let $Du = u'$, hu, δ_0, and $x^{-1}u$ be the linear forms defined by: $\langle u', f\rangle := -\langle u, f'\rangle$, $\langle hu, f\rangle := \langle u, hf\rangle$, $\langle \delta_c, f\rangle := f(c)$, and $\langle (x-c)^{-1}u, f\rangle := \langle u, \theta_c f\rangle$ where $(\theta_c f)(x) = \dfrac{f(x) - f(c)}{x - c}$, $c \in \mathbb{C}$,

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