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BLOW-UP RESULTS FOR A REACTION-DIFFUSION SYSTEM*

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Abstract. We consider the Cauchy problem for the reaction-diffusion system with the nonlinear terms $|x|^{\sigma_j} u^{p_j} v^{q_j}$. In this system, the exponents p_1 and q_2 play a crucial role to determine the behavior of the solutions. Using an ODE method, we prove the Fujita-type nonexistence results for $p_1, q_2 < 1$, for $q_2 < 1 < p_1$ or for $p_1, q_2 > 1$. Moreover, we also show the nonexistence results for large initial data.

Key words. blow-up, reaction-diffusion system, Cauchy problem.

AMS subject classifications. Primary 35K57; Secondary 35B33, 35K05, 35K45

1. Introduction. We consider the Cauchy problem for the reaction-diffusion system:

$$u_t - \Delta u = |x|^{\sigma_1} u^{p_1} v^{q_1}, \qquad x \in \boldsymbol{R}^N, \quad t > 0, \tag{1.1}$$

$$v_t - \Delta v = |x|^{\sigma_2} u^{p_2} v^{q_2}, \qquad x \in \boldsymbol{R}^N, \quad t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x) \geq 0, \not\equiv 0, \qquad x \in \boldsymbol{R}^N,$$

$$v(x,0) = v_0(x) \geq 0, \not\equiv 0, \qquad x \in \boldsymbol{R}^N,$$

where p_j, $q_j \geq 0$, $\sigma_j > \max(-2, -N)$ $(j = 1, 2)$, and $p_1, q_2 \neq 1$.

There are some papers on the Cauchy problem for semilinear reaction-diffusion systems. In [2], Escobedo and Herrero proved the existence and nonexistence of global solutions, so-called the Fujita-type result, for $\sigma_1 = \sigma_2 = p_1 = q_2 = 0$, p_2, $q_1 \geq 1$, $p_2 q_1 > 1$. As an extension of [2], Mochizuki and Huang [4] showed the Fujita-type result for $p_1 = q_2 = 0$, $0 \leq \sigma_1 < N(p_2 - 1)$, $0 \leq \sigma_2 < N(q_1 - 1)$, p_2, $q_1 \geq 1$, $p_2 q_1 > 1$. Both of the results show that the interaction between the unknown functions in the nonlinear terms determines the behavior of solutions of the system.

In [3], Escobedo and Levine showed an interesting result for $\sigma_1 = \sigma_2 = 0$, p_1, p_2, q_1, $q_2 \geq 0$. Under the assumption that $p_2 + q_2 \geq p_1 + q_1 > 0$, they showed that if $p_1 > 1$, the solutions of the system behave like a solution of the single equation $u_t - \Delta u = u^{p_1 + q_1}$.

Our aim of this paper is to show the conditions for the nonexistence of global solutions of the system (1.1) and (1.2) in three cases $p_1, q_2 < 1$, $q_2 < 1 < p_1$, or $p_1, q_2 > 1$. The conditions are about the relation between the exponents p_j, q_j, σ_j, and the initial data. See Theorems 2.1-2.3 in the next section. Comparing each part (i) in the theorems with the results in [1], we see that our conditions are optimal because the authors in [1] have proved the following results:
(i) Let $p_1 < 1$, $q_2 < 1$ and $p_2 q_1 - (1 - p_1)(1 - q_2) > 0$. If $\alpha < N/2$ and $\beta < N/2$, then global solutions exist for small initial data.
(ii) Let $p_1 > 1$ and $q_2 < 1$. If $\alpha < N/2$ and $p_1 + q_1 > 1 + (2 + \sigma_1)/N$, then global solutions exist for small initial data.
(iii) Let $p_1 > 1$ and $q_2 > 1$. If $p_1 + q_1 > 1 + (2 + \sigma_1)/N$ and $p_2 + q_2 > 1 + (2 + \sigma_2)/N$, then global solutions exist for small initial data.

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