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A LOCAL-STRUCTURE-PRESERVING LOCAL DISCONTINUOUS GALERKIN METHOD FOR THE LAPLACE EQUATION*

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Abstract. In this paper, we present a local-structure-preserving local discontinuous Galerkin (LDG) method for the Laplace equation. The method is based on the standard LDG formulation and uses piecewise harmonic polynomials, which satisfy the partial differential equation (PDE) exactly inside each element, as the approximating solutions for the primitive variable u, leading to a significant reduction of the degrees of freedom for the final system and hence the computational cost, without sacrificing the convergence quality of the solutions. An *a priori* error estimate in the energy norm is established. Numerical experiments are performed to verify optimal convergence rates of the local-structure-preserving LDG method in the energy norm and in the L^2-norm, as well as to compare it with the standard LDG method to demonstrate comparable performance of the two methods even though the new local-structure-preserving LDG method is computational less expensive.

Key words. discontinuous Galerkin method, Laplace equation, local-structure-preserving

AMS subject classifications. 65N30, 35J05

1. Introduction. It is well known that by allowing discontinuities in the finite element solution spaces, discontinuous Galerkin (DG) methods [5, 7], or the local discontinuous Galerkin (LDG) methods for partial differential equations (PDEs) containing higher than first spatial derivatives [6, 13, 14, 15], have more degrees of freedom compared with the traditional finite element methods. This is often considered as a drawback of the DG (or LDG) methods. However, these "extra" degrees of freedom may provide algorithm developers opportunities to design stable and accurate schemes by properly choosing the inter-element treatment (also called numerical fluxes). This issue has been pursued by many authors regarding different problems, see for example the review paper [7] and the special issue of the Journal of Scientific Computing on discontinuous Galerkin methods [8].

In this paper, we study a local-structure-preserving LDG method for solving the Laplace equation as a model equation, to explore another issue related to these "extra" degrees of freedom. That is, the discontinuities of the solution spaces also provide flexibility for us to choose the local solution spaces in each element, which is definitely not easy for traditional continuous finite element methods.

Our local-structure-preserving LDG method for the Laplace equation is based on the standard LDG method for elliptic equations [2]. The distinctive feature of the method is the use of harmonic polynomials (polynomials which satisfy $\triangle u = 0$) to approximate the primitive variable u in each element. In other words, the PDE is satisfied exactly in each element by the numerical solution. As a byproduct, the number of degrees of freedom for the final system is significantly reduced, therefore less computational cost is expected compared with the standard LDG method. Meanwhile, the approximation properties of the chosen spaces can guarantee no loss of the

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