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PERTURBATIONS OF NON SELF-ADJOINT STURM-LIOUVILLE PROBLEMS, WITH APPLICATIONS TO HARMONIC OSCILLATORS*

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Abstract. We study the behavior of the limit of the spectrum of a non self-adjoint Sturm-Liouville operator with analytic potential as the semi-classical parameter $h \to 0$. We get a good description of the spectrum and limit spectrum near ∞. We also study the action of one special perturbation of the operator (adding a Heaviside function), and prove that the limit spectrum is very unstable. As an illustration we describe the limit spectrum as $h \to 0$ for $P^h = -h^2\Delta + ix^2$ and the effect of this perturbation.

Key words. Eigenvalues, Non self-adjoint operators, Sturm-Liouville theory

AMS subject classifications. 34L05 34D10 34L40

1. Introduction. This paper is devoted to non self-adjoint Sturm-Liouville problems. We study the spectrum of the 1-dimensional, semiclassical Schrödinger operator on $L^2([-1,1])$ with Dirichlet boundary condition, given by

$$H^h = -h^2 \frac{d}{dx^2} + V(x). \tag{1}$$

The potential V is a complex valued function on $[-1,1]$, which extends holomorphically to some domain in $\mathbb{C}$. The boundary value at ± 1 play no special rule but are fixed to avoid more notation.

The study of such operators is motivated by the Orr-Sommerfeld equation with linear profile [6] or by the non linear Zakharov-Shabat eigenvalue problem, cf. work of Miller [13].

As an application we will focus on the case where

$$V(x) = ix^2, \tag{2}$$

and shall write P^h for the corresponding operator. From this one could also study the slightly more general case $-h^2\frac{d}{dx^2} + e^d x^2$, $d \in \mathbb{C}$ using a change of variable. The spectrum of this operator on $\mathbb{R}$ (without Dirichlet condition) was analyzed by Davies [2], cf. also the recent work of Hitrik [9].

It is well known that the spectrum of a non self-adjoint operator is unstable under perturbation of the operator. This motivates the introduction of the pseudo-spectrum, which has now been studied by many people, particularly Trefethen (who maintains the web archive

http:web.comlab.ox.ac.uk/projects/pseudospectra) and Davies [3],[4]; we note also the recent paper of Denker, Sjöstrand and Zworski [5].

We also consider the following perturbation of H, non smooth:

For $\beta \in (-1,1)$ and $\delta \geq 0$, let $H_{\delta,\beta}$

$$H^h_{\delta,\beta} = -h^2\frac{d}{dx^2} + V_{\delta,\beta}(x), \quad V_{\delta,\beta}(x) = \begin{cases} V(x) + i\delta, & x > \beta \\ V(x) - i\delta, & x < \beta \end{cases}, \tag{3}$$

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