METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 13, No. 1, pp. 063–088, March 2006

004

THE CHEN-RUBIN CONJECTURE IN A CONTINUOUS SETTING*

CHRISTIAN BERG† AND HENRIK L. PEDERSEN‡

Abstract. We study the median $m(x)$ in the gamma distribution with parameter x and show that $0 < m'(x) < 1$ for all $x > 0$. This proves a generalization of a conjecture of Chen and Rubin from 1986: The sequence $m(n) - n$ decreases for $n \geq 1$. We also describe the asymptotic behaviour of m and m' at zero and at infinity.

Key words. median, gamma function, gamma distribution

AMS subject classifications. primary 60E05; secondary 41A60, 33B15

1. Introduction. The gamma distribution with (positive) parameter x has density with respect to Lebesgue measure on $(0, \infty)$ given by $e^{-t}t^{x-1}/\Gamma(x)$. The median $m(x)$ of this distribution is defined implicitly as

$$\int_0^{m(x)} \frac{e^{-t}t^{x-1}}{\Gamma(x)}\, dt = \frac{1}{2},$$

or

$$\int_0^{m(x)} e^{-t}t^{x-1}\, dt = \frac{1}{2}\int_0^{\infty} e^{-t}t^{x-1}\, dt. \tag{1}$$

This is of course equivalent to

$$\int_{m(x)}^{\infty} e^{-t}t^{x-1}\, dt = \frac{1}{2}\int_0^{\infty} e^{-t}t^{x-1}\, dt. \tag{2}$$

We show that m is continuous and increasing. This is a consequence of a result about general convolution semigroups of probabilities on the positive half-line, that is given in Section 2. There we also show that m is real analytic and that m satisfies a certain differential equation.

We shall mainly study m through the function

$$\varphi(x) \equiv \log \frac{x}{m(x)}, \quad x > 0. \tag{3}$$

This function appears if we make the substitution $u = \log(x/t)$ in the relation (2):

$$\int_{-\infty}^{\varphi(x)} e^{-x(e^{-u}+u)}\, du = \frac{1}{2}\int_{-\infty}^{\infty} e^{-x(e^{-u}+u)}\, du. \tag{4}$$

Chen and Rubin (see [7]) studied the median of the gamma distribution and proved that $x - 1/3 < m(x) < x$ for $x > 0$. The relation (4) was also used in [7] to

*Received October 29, 2004; accepted for publication August 24, 2006.

†Department of Mathematics, University of Copenhagen, Universitetsparken 5, DK-2100, Copenhagen, Denmark (berg@math.ku.dk).

‡Department of Natural Sciences, Royal Veterinary and Agricultural University, Thorvaldsensvej 40, DK-1871, Copenhagen, Denmark (henrikp@dina.kvl.dk). Research supported by the Carlsberg Foundation.