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AN INEQUALITY OF HADAMARD TYPE FOR PERMANENTS*

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Abstract. Let F be an $N \times N$ complex matrix whose jth column is the vector $\vec{f}_j$ in $\mathbb{C}^N$. Let $|\vec{f}_j|^2$ denote the sum of the absolute squares of the entries of $\vec{f}_j$. Hadamard's inequality for determinants states that $|\det(F)| \le \prod_{j=1}^N |\vec{f}_j|$. Here we prove a sharp upper bound on the permanent of F, which is $|\mathrm{perm}(F)| \le \frac{N!}{N^{N/2}} \prod_{j=1}^{N} |\vec{f}_j|$, and we determine all of the cases of equality.

We also discuss the case in which $|\vec{f}_j|$ is replaced by the ℓ_p norm of the vector $\vec{f}$ considered as a function on $\{1, 2, \ldots, N\}$. We note a simple sharp inequality for $p = 1$, and obtain bounds for intermediate p by interpolation.

Key words. Permanent, Hadamard inequality, heat kernel

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1. Introduction.

Let F be an $N \times N$ complex matrix whose jth column is the vector $\vec{f}_j$ in $\mathbb{C}^N$. Let $|\vec{f}_j|^2$ denote the sum of the absolute squares of the entries of $\vec{f}_j$. Hadamard's inequality for determinants [4] states that $|\det(F)| \le \prod_{j=1}^N |\vec{f}_j|$. Here we prove a sharp upper bound on the permanent of F:

THEOREM 1.1. *For any vectors $\vec{f}_1, \ldots, \vec{f}_N$ in $\mathbb{C}^N$ we have the inequality*

$$|\mathrm{perm}(F)| \le \frac{N!}{N^{N/2}} \prod_{j=1}^{N} |\vec{f}_j| \ . \tag{1.1}$$

For $N > 2$, there is equality in (1.1) if and only if at least one of the vectors $\vec{f}_j$ is zero, or else F is a rank one matrix and, moreover, each of the vectors $\vec{f}_j$ is a constant modulus vector; i.e., its entries all have the same absolute value.

The conditions for equality can be reformulated as follows: There is equality in (1.1) if and only if one or more of the vectors $\vec{f}_j$ is zero, or else there are numbers r_j, ξ_j, ζ_j, $j = 1, \ldots, N$, with each $r_j > 0$ and each ξ_j and ζ_j lying on the unit circle in the complex plane, so that

$$F_{j,k} = \xi_j \zeta_k r_k$$

for each j, k.

We shall give two proofs of this inequality. The first turns on recognizing (1.1) as a close relative of the Brascamp–Lieb type inequality that we recently proved [2] for integrals of products of functions on the sphere $\mathbb{S}^{N-1}$. To explain this way of viewing (1.1), we first introduce some notation and terminology.

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