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AN OPTIMAL TRANSPORTATION METRIC FOR SOLUTIONS OF THE CAMASSA-HOLM EQUATION*

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Dedicated to Prof. Joel Smoller in the occasion of his 70-th birthday

Abstract. In this paper we construct a global, continuous flow of solutions to the Camassa-Holm equation on the entire space H^1. Our solutions are conservative, in the sense that the total energy $\int(u^2 + u_x^2)\,dx$ remains a.e. constant in time. Our new approach is based on a distance functional $J(u,v)$, defined in terms of an optimal transportation problem, which satisfies $\frac{d}{dt}J(u(t),v(t)) \leq \kappa \cdot J(u(t),v(t))$ for every couple of solutions. Using this new distance functional, we can construct arbitrary solutions as the uniform limit of multi-peakon solutions, and prove a general uniqueness result.

Key words. Camassa-Holm equation, optimal transportation metric, conservative solution

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1. Introduction.

The Camassa-Holm equation can be written as a scalar conservation law with an additional integro-differential term:

$$u_t + (u^2/2)_x + P_x = 0\,, \tag{1.1}$$

where P is defined as a convolution:

$$P \doteq \frac{1}{2}e^{-|x|} * \left(u^2 + \frac{u_x^2}{2}\right). \tag{1.2}$$

For the physical motivations of this equation we refer to [CH], [CM1], [CM2], [J]. Earlier results on the existence and uniqueness of solutions can be found in [XZ1], [XZ2]. One can regard (1.1) as an evolution equation on a space of absolutely continuous functions with derivatives $u_x \in \mathbf{L}^2$. In the smooth case, differentiating (1.1) w.r.t. x one obtains

$$u_{xt} + uu_{xx} + u_x^2 - \left(u^2 + \frac{u_x^2}{2}\right) + P = 0\,. \tag{1.3}$$

Multiplying (1.1) by u and (1.3) by u_x we obtain the two conservation laws with source term

$$\left(\frac{u^2}{2}\right)_t + \left(\frac{u^3}{3} + u\,P\right)_x = u_x\,P\,, \tag{1.4}$$

$$\left(\frac{u_x^2}{2}\right)_t + \left(\frac{uu_x^2}{2} - \frac{u^3}{3}\right)_x = -u_x\,P\,. \tag{1.5}$$

As a consequence, for regular solutions the total energy

$$E(t) \doteq \int \left[u^2(t,x) + u_x^2(t,x)\right]dx \tag{1.6}$$

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