JOURNAL OF
SYMPLECTIC GEOMETRY
Volume 9, Number 3, 403–428, 2011

# ON COHOMOLOGICAL DECOMPOSITION OF ALMOST-COMPLEX MANIFOLDS AND DEFORMATIONS

Daniele Angella and Adriano Tomassini

While small deformations of compact Kähler manifolds are Kähler too, we prove that the cohomological property to be $\mathcal{C}^\infty$-pure-and-full is not a stable condition under small deformations. This property, which has been recently introduced and studied by Li and Zhang in [**24**] and Draghici *et al.* in [**13**, **14**], is weaker than the Kähler one and characterizes the almost-complex structures inducing a decomposition in cohomology. We also study the stability of this property along curves of almost-complex structures constructed starting from the harmonic representatives in special cohomology classes.

## 1. Introduction

Let $(M, J)$ be a compact almost-complex $2n$-dimensional manifold and let $\omega$ be a symplectic form on $M$. Then $J$ is said to be $\omega$-*tamed* if $\omega(\cdot, J\cdot) > 0$ and $\omega$-*compatible* (or $\omega$-*calibrated*) if $g(\cdot,\cdot) := \omega(\cdot, J\cdot)$ is a $J$-Hermitian metric. Define the *tamed cone* $\mathcal{K}_J^{\mathrm{t}}$ as the open convex cone given by the projection in cohomology of the space of the symplectic forms taming $J$, namely

$$\mathcal{K}_J^{\mathrm{t}} \stackrel{\mathrm{def}}{=} \left\{[\omega] \in H^2_{dR}(M;\mathbb{R}) \mid J \text{ is } \omega - \text{tamed}\right\},$$

and the *compatible cone* $\mathcal{K}_J^{\mathrm{c}}$ as its subcone given by the projection of the space of the symplectic forms compatible with $J$, namely

$$\mathcal{K}_J^{\mathrm{c}} \stackrel{\mathrm{def}}{=} \left\{[\omega] \in H^2_{dR}(M;\mathbb{R}) \mid J \text{ is } \omega - \text{compatible}\right\}.$$

Li and Zhang proved in [**24**, Corollay 3.2] that if $J$ is integrable and $\mathcal{K}_J^{\mathrm{c}}$ is non-empty then the following relation between the two cones holds:

$$\mathcal{K}_J^{\mathrm{t}} = \mathcal{K}_J^{\mathrm{c}} + \left(\left(H^{2,0}_{\overline{\partial}}(M) \oplus H^{0,2}_{\overline{\partial}}(M)\right) \cap H^2_{dR}(M;\mathbb{R})\right); \tag{1.1}$$

they also proved (see [**24**, Theorem 1.2]) that, given a complex compact surface $(M, J)$, if there is a symplectic structure $\omega$ such that $J$ is $\omega$-tamed then $(M, J)$ admits a Kähler structure (see also [**29**, Proposition 1.6]), i.e.