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LAGRANGIAN FLOER HOMOLOGY OF THE CLIFFORD TORUS AND REAL PROJECTIVE SPACE IN ODD DIMENSIONS

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The Floer homology of the pair $(\mathbb{R}P^{2n-1}, T^{2n-1})$ in $\mathbb{C}P^{2n-1}$ is calculated and is shown to have dimension 2^n, both with $\mathbb{Z}_2$ and $\Lambda_{\mathbb{Z}_2}$ coefficients. In particular, this implies that $\mathbb{R}P^{2n-1}$ and T^{2n-1} must always intersect in at least 2^n points under Hamiltonian isotopy (if the intersection is transverse).

1. Introduction

The Clifford torus and real projective space are both Lagrangian submanifolds of complex projective space. The Clifford torus is

$$T^n = \{\ [z_0 : \cdots : z_n] \in \mathbb{C}P^n \mid |z_0| = |z_1| = \cdots = |z_n|\ \},$$

and real projective space is

$$\mathbb{R}P^n = \{\ [z_0 : \cdots : z_n] \in \mathbb{C}P^n \mid z_i \in \mathbb{R}\ \}.$$

They intersect in the 2^n points $[\pm 1 : \cdots : \pm 1]$. Two interesting questions to ask are the following: Can they be disjoined from each other by Hamiltonian isotopy? If not, what is the minimum number of points that they must intersect in? In this article we use Lagrangian Floer homology to investigate these questions. The main result is

Theorem 1.1.

$$\mathrm{HF}(\mathbb{R}P^{2n-1}, T^{2n-1} : \mathbb{Z}_2) = (\mathbb{Z}_2)^{2^n}.$$

The Floer chain group is generated by the intersection points of the submanifolds and the homology is invariant under Hamiltonian isotopy. Therefore Theorem 1.1 immediately implies

Theorem 1.2. *If ϕ is a Hamiltonian diffeomorphism and $\mathbb{R}P^{2n-1}$ intersects $\phi(T^{2n-1})$ transversely, then the intersection contains at least 2^n points.*