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COMPLEXIFICATIONS OF MORSE FUNCTIONS AND THE DIRECTED DONALDSON–FUKAYA CATEGORY

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Let N be a closed four-dimensional manifold which admits a self-indexing Morse function $f : N \longrightarrow \mathbb{R}$ with only three critical values $0, 2, 4$, and a unique maximum and minimum. Let g be a Riemannian metric on N such that (f, g) is Morse–Smale. We construct from (N, f, g) a certain six-dimensional exact symplectic manifold M, together with some exact Lagrangian spheres V_4, V_2^j, V_0 in M, $j = 1, \ldots, k$. These spheres correspond to the critical points x_4, x_2^j, x_0 of f, where the subscript indicates the Morse index. (In a companion paper we explain how $(M, V_4, \{V_2^j\}, V_0)$ is a model for the regular fiber and vanishing spheres of the complexification of f, viewed as a Lefschetz fibration on the disk cotangent bundle $D(T^*N)$.) Our main result is a computation of the Lagrangian Floer homology groups

$$\mathrm{HF}(V_4, V_2^j),\ \mathrm{HF}(V_2^j, V_0),\ \mathrm{HF}(V_4, V_0)$$

and the triangle product

$$\mu_2 : \mathrm{HF}(V_4, V_2^j) \otimes \mathrm{HF}(V_2^j, V_0) \longrightarrow \mathrm{HF}(V_4, V_0)$$

in terms of the Morse theory of (N, f, g). The outcome is that the directed Donaldson–Fukaya category of $(M, V_4, \{V_2^j\}, V_0)$ is isomorphic to the flow category of (N, f, g).

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