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BERNSTEIN POLYNOMIALS, BERGMAN KERNELS AND TORIC KÄHLER VARIETIES

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We show that the classical Bernstein polynomials $B_N(f)(x)$ on the interval $[0,1]$ (and their higher dimensional generalizations on the simplex $\Sigma_m \subset \mathbb{R}^m$) may be expressed in terms of Bergman kernels for the Fubini–Study metric on $\mathbb{CP}^m$: $B_N(f)(x)$ is obtained by applying the Toeplitz operator $f(N^{-1}D_\theta)$ to the Fubini–Study Bergman kernels. The expression generalizes immediately to any toric Kähler variety and Delzant polytope, and gives a novel definition of Bernstein "polynomials" $B_{h^N}(f)$ relative to any toric Kähler variety. They uniformly approximate any continuous function f on the associated polytope P with all the properties of classical Bernstein polynomials. Upon integration over the polytope, one obtains a complete asymptotic expansion for the Dedekind–Riemann sums $\frac{1}{N^m}\sum_{\alpha \in NP} f(\frac{\alpha}{N})$ of $f \in C^\infty(\mathbb{R}^m)$, of a type similar to the Euler–MacLaurin formulae.

1. Introduction

Our starting point is the observation that the classical Bernstein polynomials

$$B_N(f)(x) = \sum_{\alpha \in \mathbb{N}^m : |\alpha| \le N} \binom{N}{\alpha} x^\alpha (1 - \|x\|)^{N-|\alpha|} f\left(\frac{\alpha}{N}\right), \tag{1.1}$$

on the m-simplex $\Sigma_m \subset \mathbb{R}^m$ may be expressed in terms of the Bergman–Szegö kernels $\Pi_{h_{\mathrm{FS}}^N}(z,w)$ for the Fubini–Study metric on $\mathbb{CP}^m$: Let $e^{i\theta}$ denote the standard $\mathbf{T}^m = (S^1)^m$ action on $\mathbb{C}^m$ and and let D_{θ_j} denote the linearization (or "quantization") of its infinitesimal generators on $H^0(\mathbb{CP}^m, \mathcal{O}(N))$. As will be shown in Section 2 (see also Section 4),

$$B_N(f)(x) = \frac{1}{\Pi_{h_{\mathrm{FS}}^N}(z,z)} f(N^{-1}D_\theta)\Pi_{h_{\mathrm{FS}}^N}(e^{i\theta}z,z)|_{\theta=0, z=\mu_{h_{\mathrm{FS}}}^{-1}(x)}, \tag{1.2}$$

where $f \in C_0^\infty(\mathbb{R}^m)$. Here, $\Pi_{h_{\mathrm{FS}}^N}$ denotes the Bergman–Szegö kernel on powers $\mathcal{O}(N) \to \mathbb{CP}^m$ of the invariant hyperplane line bundle, $f(N^{-1}D_\theta)$ is