JOURNAL OF
SYMPLECTIC GEOMETRY
Volume 6, Number 1, 33–59, 2008

COVERING SPACES AND $\mathbb{Q}$-GRADINGS ON HEEGAARD FLOER HOMOLOGY

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Heegaard Floer homology, first introduced by P. Ozsváth and Z. Szabó in [**OS04b**], associates to a 3-manifold Y, a family of relatively graded abelian groups $HF(Y,\mathfrak{t})$, indexed by Spin^c structures $\mathfrak{t}$ on Y. In the case that Y is a rational homology sphere, Ozsváth and Szabó lift the relative $\mathbb{Z}$-grading to an absolute $\mathbb{Q}$-grading [**OS06**]. This induces a relative $\mathbb{Q}$-grading on $\bigoplus_{\mathfrak{t}\in\mathrm{Spin}^c(Y)} HF(Y,\mathfrak{t})$. In this paper, we describe an alternate construction of this relative $\mathbb{Q}$-grading by studying the Heegaard Floer homology of covering spaces.

1. Introduction

In [**OS04b**], P. Oszváth and Z. Szabó associated to a 3-manifold Y families $\widehat{HF}(Y,\mathfrak{t})$, $HF^+(Y,\mathfrak{t})$, $HF^-(Y,\mathfrak{t})$, and $HF^\infty(Y,\mathfrak{t})$ of abelian groups, indexed by Spin^c structures $\mathfrak{t}$ on Y, collectively known as Heegaard Floer homology. (Below, we shall use $HF(Y,\mathfrak{t})$ to refer to any of these groups.) These groups arise as the homology groups of certain Lagrangian intersection Floer chain complexes $\widehat{CF}(Y,\mathfrak{t})$, $CF^+(Y,\mathfrak{t})$, $CF^-(Y,\mathfrak{t})$, and $CF^\infty(Y,\mathfrak{t})$, and as such inherit relative $\mathbb{Z}$-gradings or, if $c_1(\mathfrak{t})$ is non-torsion, relative $\mathbb{Z}/n$ gradings; we denote all of these gradings by gr.

In [**OS06**], for $c_1(\mathfrak{t})$ torsion, Ozsváth and Szabó used a bordism construction to lift gr to an absolute $\mathbb{Q}$-grading on $HF(Y,\mathfrak{t})$. In other words, they found an absolute $\mathbb{Q}$-grading $\widetilde{\mathrm{gr}}$ on $HF(Y,\mathfrak{t})$ satisfying $\mathrm{gr}(\xi,\eta) = \widetilde{\mathrm{gr}}(\xi) - \widetilde{\mathrm{gr}}(\eta)$ for all homogeneous elements $\xi,\eta \in HF(Y,\mathfrak{t})$. This defines an absolute $\mathbb{Q}$-grading on the group

$$HF(Y,\text{torsion}) := \bigoplus_{c_1(\mathfrak{t})\text{ is torsion}} HF(Y,\mathfrak{t}).$$

In [**OS03**], they used this absolute $\mathbb{Q}$-grading to give restrictions on which knots can, under surgery, give rise to lens spaces, and to give restrictions