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TEST CONFIGURATIONS FOR K-STABILITY AND GEODESIC RAYS

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Let X be a compact complex manifold, $L \to X$ an ample line bundle over X, and $\mathcal{H}$ the space of all positively curved metrics on L. We show that a pair (h_0, T) consisting of a point $h_0 \in \mathcal{H}$ and a test configuration $T = (\mathcal{L} \to \mathcal{X} \to \mathbf{C})$, canonically determines a weak geodesic ray $R(h_0, T)$ in $\mathcal{H}$ which emanates from h_0. Thus a test configuration behaves like a vector field on the space of Kähler potentials $\mathcal{H}$. We prove that R is non-trivial if the $\mathbf{C}^\times$ action on X_0, the central fiber of $\mathcal{X}$, is non-trivial. The ray R is obtained as limit of smooth geodesic rays $R_k \subseteq \mathcal{H}_k$, where $\mathcal{H}_k \subseteq \mathcal{H}$ is the subspace of Bergman metrics.

Dedicated to Dusa McDuff

1. Introduction

Let X be a compact complex manifold. According to a basic conjecture of Yau [**33**], the existence of canonical metrics on X should be equivalent to a stability condition in the sense of geometric invariant theory. A version of this conjecture, due to Tian [**31**] and Donaldson [**14**], says that if $L \to X$ is an ample line bundle, then X has a metric of constant scalar curvature in $c_1(L)$ if and only if the pair (X, L) is K-stable, that is, if and only if the Futaki invariant $F(T)$ is negative for each non-trivial test configuration T. In particular, $F(T) < 0$ for all such T should imply that the K-energy $\nu : \mathcal{H} \to \mathbf{R}$ is bounded below, where $\mathcal{H}$ is the space of all positively curved metrics on L.

Now it is well known that the K-energy is convex along geodesics of $\mathcal{H}$ (Donaldson [**12**]). Thus, if $h_0 \in \mathcal{H}$ and if $R : (-\infty, 0] \to \mathcal{H}$ is a smooth geodesic ray emanating from h_0, then the restriction of ν to R is a smooth convex function $\nu_R : (-\infty, 0] \to \mathbf{R}$ and hence $\lim_{t\to-\infty} \dot\nu_R = a(R)$ is well defined (here $\dot\nu_R$ is the time derivative of the K-energy). In particular, if $a(R) < 0$, then ν is bounded below on the ray R.